

Multivariable and Vector Calculus

Alvin Lin

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Review 2

Arc Length:

$$L = \int_a^b |\vec{v}'(t)| dt$$

Partial Derivative Chain Rules:

$$z = f(x, y), \quad x = x(t), \quad y = y(t), \quad \frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}$$

$$F(x, y, z) = 0, \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

Gradient Vector:

$$\vec{\nabla} F = \langle F_x, F_y, F_z \rangle \perp \text{surface}$$

$$D_{\vec{u}} f = \vec{\nabla} f \cdot \vec{u}, \quad |\vec{u}| = 1$$

$$\max D_{\vec{u}} f = |\vec{\nabla} f| \text{ in } \vec{u} = \frac{\vec{\nabla} f}{|\vec{\nabla} f|}$$

Linear approximation:

$$\Delta f \approx df$$

$$f(x, y) \approx f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0)$$

Practice Problem

Given $\overrightarrow{v(t)}$ for $C = S_1 \cap S_2$:

$$S_1 : z = \sqrt{x^2 + y^2}$$

$$S_2 : z = 1 + y$$

$$\sqrt{x^2 + y^2} = 1 + y$$

$$x^2 + y^2 = 1 + 2y + y^2$$

$$x^2 = 1 + 2y$$

$$y = \frac{x^2 - 1}{2}$$

$$\overrightarrow{r(t)} = \left\langle t, \frac{t^2 - 1}{2}, 1 + t \right\rangle$$

Practice Problem

$$C : \overrightarrow{v(t)} = \langle 2 \cos(t), 2 \sin(t), e^t \rangle$$

Find a point on C such that the line tangent to the curve is parallel to the plane described by $\sqrt{3}x + y = 1$.

The line tangent and the normal vector of the plane must have a dot product of 0 if they are parallel.

$$\begin{aligned} \overrightarrow{v'(t)} \cdot \langle \sqrt{3}, 1, 0 \rangle &= 0 \\ &\equiv \langle -2 \sin(t), 2 \cos(t), e^t \rangle \cdot \langle \sqrt{3}, 1, 3 \rangle = 0 \\ &\equiv -2\sqrt{3} \sin(t) + 2 \cos(t) = 0 \\ &\equiv 2 \cos(t) = 2\sqrt{3} \sin(t) \\ &\equiv \tan(t) = \frac{\sqrt{3}}{3} \end{aligned}$$

We can simply plug t back into the vector equation to find the point.

Practice Problem

Find the angle at the point of intersection between the curves:

$$C_1 : \overrightarrow{v_1(t)} = \langle t, 1 - t, 3 + t^2 \rangle$$

$$C_2 : \overrightarrow{v_2(\tau)} = \langle 3 - \tau, \tau - 2, \tau^2 \rangle$$

$$t = 3 - \tau$$

$$1 - t = \tau - 2$$

$$3 + t^2 = \tau^2$$

$$1 - 3 + \tau = \tau - 2$$

$$3 + 9 - 6\tau + \tau^2 = \tau^2$$

$$\tau = 2$$

$$t = 1$$

$$\overrightarrow{v_1(1)} = \langle 1, -1, 2t \rangle = \langle 1, -1, 2 \rangle$$

$$\overrightarrow{v_2(2)} = \langle -1, 1, 2\tau \rangle = \langle -1, 1, 4 \rangle$$

$$\cos \theta = \frac{\langle 1, -1, 2 \rangle \cdot \langle -1, 1, 4 \rangle}{\sqrt{6}\sqrt{18}}$$

$$= \frac{6}{\sqrt{6}\sqrt{18}}$$

$$= \frac{\sqrt{6}}{\sqrt{18}}$$

$$= \frac{\sqrt{3}}{3}$$

Practice Problem

Check that an object moving with constant speed has an acceleration vector that is perpendicular to the velocity vector.

$$\begin{aligned}|\vec{v}(t)| &= c \\ |\vec{v}(t)|^2 &= c^2 \\ \vec{v}(t) \cdot \vec{v}(t) &= c^2\end{aligned}$$

Derive in terms of t

$$\begin{aligned}\vec{v}'(t) \cdot \vec{v}(t) + \vec{v}(t) \cdot \vec{v}'(t) &= 0 \\ 2(\vec{v}(t) \cdot \vec{v}'(t)) &= 0 \\ \therefore \vec{v}(t) &\perp \vec{v}'(t)\end{aligned}$$

Practice Problem

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^y}{2x^4 - 5y^2}$$

Suppose $y = kx$:

$$\lim_{(x,kx) \rightarrow (0,0)} \frac{x^2 kx}{2x^4 - 5k^2 x^2} = \lim \frac{kx}{2x^2 - 5k^2} = 0$$

Suppose $y = kx^2$:

$$\lim_{(x,kx^2) \rightarrow (0,0)} \frac{x^2 kx^2}{2x^4 - 5k^2 x^4} = \frac{k}{2 - 5k^2}$$

Since the limit is different on different curves approaching the point, the limit does not exist.

Practice Problem

A sand pile is in a conical shape with radius 10 and height 30. It starts to rain, causing the height to decrease at a rate of -2 units per minute. This causes the sand pile to flatten, but the volume remains the same. What is the rate of change of the

radius?

$$\begin{aligned}r &= 10 \\h &= 30 \\\frac{dh}{dt} &= -2 \\\frac{dV}{dt} &= 0 \\V &= \frac{1}{3}\pi r^2 h \\dV &= \frac{\partial V}{\partial r} \frac{dr}{dt} + \frac{\partial V}{\partial h} \frac{dh}{dt} \\0 &= \frac{2}{3}\pi r h \frac{dr}{dt} + \frac{1}{3}\pi r^2 \frac{dh}{dt} \\0 &= \frac{2}{3}\pi(10)(30) \frac{dr}{dt} + \frac{1}{3}\pi(10^2)(-2) \\\frac{dr}{dt} &= \frac{\frac{200\pi}{3}}{\frac{600\pi}{3}} = \frac{1}{3}\end{aligned}$$

Suppose our measurements were inaccurate and our radius measurement is $10 \pm 1\%$ and our height measurement is $30 \pm 2\%$. Give the approximate max change in the volume.

$$\begin{aligned}r &= 10 \\h &= 30 \\dr &= \pm \frac{1}{10} \\dh &= \pm \frac{6}{10} \\\Delta v &= dV \\&= \frac{\partial V}{\partial r} dr + \frac{\partial V}{\partial h} dh \\&= \frac{2}{3}\pi r h dr + \frac{\pi}{3} r^2 dh \\&= \frac{2}{3}\pi(300)(\pm \frac{1}{10}) - \frac{\pi}{3}(100)(\pm \frac{6}{10})\end{aligned}$$

At the ends of the ranges, the maximum change in volume based on the error is $20\pi + 20\pi = 40\pi$.

Practice Problem

Find $\frac{dz}{dx}$ given the curve:

$$yz + x \ln(y) - z^2 = 0$$
$$\frac{dz}{dx} = -\frac{F_x}{F_z} = -\frac{\ln(y)}{y - 2z}$$

Practice Problem

Find the line tangent to the curve $C = S_1 \cap S_2$ at the point $(3,4,5)$:

$$S_1 : z = \sqrt{x^2 + y^2}$$

$$S_2 : z = 1 + y$$

$$\begin{aligned} \vec{u} &= \overrightarrow{\nabla F_1} \times \overrightarrow{\nabla F_2} \\ &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{-x}{\sqrt{x^2+y^2}} & \frac{-y}{\sqrt{x^2+y^2}} & 1 \\ 0 & -1 & 1 \end{vmatrix} \\ &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\frac{3}{5} & -\frac{4}{5} & 1 \\ 0 & -1 & 1 \end{vmatrix} \\ &= \left\langle \frac{1}{5}, \frac{3}{5}, \frac{3}{5} \right\rangle \parallel \langle 1, 3, 3 \rangle \\ l_{\text{tangent}} &= \begin{cases} x = 3 + t \\ y = 4 + 3t \\ z = 5 + 3t \end{cases} \end{aligned}$$

Practice Problem

Evaluate the directional derivative when the vector \vec{u} goes from $(2,1)$ to $(5,5)$.

$$D_{\vec{u}}(x^2 + xy^3)(2,1)$$

$$\begin{aligned}
D_{\vec{u}}(x^2 + xy^3)(2, 1) &= \vec{\nabla} f \cdot \vec{u} \\
&= \langle 2x + y^3, 2xy^2 \rangle \cdot \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle \\
&= \langle 5, 6 \rangle \cdot \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle \\
&= \frac{39}{5}
\end{aligned}$$

Practice Problem

Approximate the value of $(0.9)^2(3.9)^2$:

We can take $(1, 4)$ as a reference point using the curve x^2y^2 .

$$\begin{aligned}
x^2y^2 &\approx 16 + (2xy^2)(x - 1) + (2x^2y)(y - 4) \\
&\approx 16 + 32(x - 1) + 8(y - 4) \\
(0.9)^2(3.9)^2 &\approx 16 + 32(-0.1) + 8(-0.1) \\
&\approx 16 - 3.2 - 0.8 \\
&\approx 12
\end{aligned}$$

You can find all my notes at <http://omgimanagerd.tech/notes>. If you have any questions, comments, or concerns, please contact me at alvin@omgimanagerd.tech