Multivariable and Vector Calculus

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Review 2

Arc Length:

$$L = \int_a^b |\overrightarrow{v'(t)}| \, \mathrm{d}t$$

Partial Derivative Chain Rules:

$$z = f(x, y), \quad x = x(t), \quad y = y(t), \quad \frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}$$

$$F(x, y, z) = 0, \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

Gradient Vector:

$$\overrightarrow{\nabla F} = \langle F_x, F_y, F_z \rangle \perp \text{surface}$$

$$D_{\vec{u}} f = \overrightarrow{\nabla f} \cdot \vec{u}, \quad |\vec{u}| = 1$$

$$\max\ D_{\vec{u}} f = |\overrightarrow{\nabla f}|\ in\ \vec{u} = \frac{\overrightarrow{\nabla f}}{|\overrightarrow{\nabla f}|}$$

Linear approximation:

$$\Delta f \approx \ \mathrm{d}f$$

$$f(x, y) \approx f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0)$$

Practice Problem

Given $\overrightarrow{v(t)}$ for $C = S_1 \cap S_2$:

$$S_1 : z = \sqrt{x^2 + y^2}$$
$$S_2 : z = 1 + y$$

$$\begin{aligned}
\sqrt{x^2 + y^2} &= 1 + y \\
x^2 + y^2 &= 1 + 2y + y^2 \\
x^2 &= 1 + 2y \\
y &= \frac{x^2 - 1}{2} \\
\overrightarrow{r(t)} &= \langle t, \frac{t^2 - 1}{2}, 1 + t \rangle
\end{aligned}$$

Practice Problem

$$C : \overrightarrow{v(t)} = \langle 2\cos(t), 2\sin(t), \mathrm{e}^t \rangle$$

Find a point on C such that the line tangent to the curve is parallel to the plane described by $\sqrt{3}x + y = 1$.
The line tangent and the normal vector of the plane must have a dot product of 0 if they are parallel.

$$\begin{aligned}
\overrightarrow{v'(t)} \cdot \langle \sqrt{3}, 1, 0 \rangle &= 0 \\
&\equiv \langle -2\sin(t), 2\cos(t), \mathrm{e}^t \rangle \cdot \langle \sqrt{3}, 1, 3 \rangle = 0 \\
&\equiv -2\sqrt{3}\sin(t) + 2\cos(t) = 0 \\
&\equiv 2\cos(t) = 2\sqrt{3}\sin(t) \\
&\equiv \tan(t) = \frac{\sqrt{3}}{3}
\end{aligned}$$

We can simply plug t back into the vector equation to find the point.

Practice Problem

Find the angle at the point of intersection between the curves:

$$C_1 : \overrightarrow{v_1(t)} = \langle t, 1 - t, 3 + t^2 \rangle$$

$$C_2 : \overrightarrow{v_2(\tau)} = \langle 3-\tau, \tau-2, \tau^2 \rangle$$

$$\begin{aligned}
t &= 3-\tau \\
1-t &= \tau-2 \\
3+t^2 &= \tau^2 \\
1-3+\tau &= \tau-2 \\
3+9-6\tau+\tau^2 &= \tau^2 \\
\tau &= 2 \\
t &= 1
\end{aligned}$$

$$\begin{aligned}
\overrightarrow{v_1(1)} &= \langle 1, -1, 2t \rangle = \langle 1, -1, 2 \rangle \\
\overrightarrow{v_2(2)} &= \langle -1, 1, 2\tau \rangle = \langle -1, 1, 4 \rangle \\
\cos\theta &= \frac{\langle 1, -1, 2 \rangle \cdot \langle -1, 1, 4 \rangle}{\sqrt{6}\sqrt{18}} \\
&= \frac{6}{\sqrt{6}\sqrt{18}} \\
&= \frac{\sqrt{6}}{\sqrt{18}} \\
&= \frac{\sqrt{3}}{3}
\end{aligned}$$

Practice Problem

Check that an object moving with constant speed has an acceleration vector that is perpendicular to the velocity vector.

$$
\begin{aligned}
|\overrightarrow{v(t)}| &= c \\
|\overrightarrow{v(t)}|^2 &= c^2 \\
\overrightarrow{v(t)} \cdot \overrightarrow{v(t)} &= c^2 \\
&\text{Derive in terms of t} \\
\overrightarrow{v'(t)} \cdot \overrightarrow{v(t)} + \overrightarrow{v(t)} \cdot \overrightarrow{v'(t)} &= 0 \\
2\left(\overrightarrow{v(t)} \cdot \overrightarrow{v'(t)}\right) &= 0 \\
\therefore\ \overrightarrow{v(t)} &\perp \overrightarrow{v'(t)}
\end{aligned}
$$

Practice Problem

$$\lim_{(x,y)\to(0,0)} \frac{x^y}{2x^4 - 5y^2}$$

Suppose $y = kx$:

$$\lim_{(x,kx)\to(0,0)} \frac{x^2kx}{2x^4 - 5k^2x^2} = \lim \frac{kx}{2x^2 - 5k^2} = 0$$

Suppose $y = kx^2$:

$$\lim_{(x,kx^2)\to(0,0)} \frac{x^2kx^2}{2x^4 - 5k^2x^4} = \frac{k}{2 - 5k^2}$$

Since the limit is different on different curves approaching the point, the limit does not exist.

Practice Problem

A sand pile is in a conical shape with radius 10 and height 30. It starts to rain, causing the height to decrease at a rate of -2 units per minute. This causes the sand pile to flatten, but the volume remains the same. What is the rate of change of the radius?

$$\begin{aligned}
r &= 10 \\
h &= 30 \\
\frac{\mathrm{d}h}{\mathrm{d}t} &= -2 \\
\frac{\mathrm{d}V}{\mathrm{d}t} &= 0 \\
V &= \frac{1}{3}\pi r^2 h \\
\mathrm{d}Vt &= \frac{\partial V}{\partial t}\frac{\mathrm{d}r}{\mathrm{d}t} + \frac{\partial V}{\partial h}\frac{\mathrm{d}h}{\mathrm{d}t} \\
0 &= \frac{2}{3}\pi r h\frac{\mathrm{d}r}{\mathrm{d}t} + \frac{1}{3}\pi r^2\frac{\mathrm{d}h}{\mathrm{d}t} \\
0 &= \frac{2}{3}\pi(10)(30)\frac{\mathrm{d}r}{\mathrm{d}t} + \frac{1}{3}\pi(10^2)(-2) \\
\frac{\mathrm{d}r}{\mathrm{d}t} &= \frac{\frac{200\pi}{3}}{\frac{600\pi}{3}} = \frac{1}{3}
\end{aligned}$$

Suppose our measurements were inaccurate and our radius measurement is $10 \pm 1\%$ and our height measurement is $30 \pm 2\%$. Give the approximate max change in the volume.

$$\begin{aligned}
r &= 10 \\
h &= 30 \\
\mathrm{d}r &= \pm\frac{1}{10} \\
\mathrm{d}h &= \pm\frac{6}{10} \\
\Delta v &= \ \mathrm{d}V \\
&= \frac{\partial V}{\partial r}\ \mathrm{d}r + \frac{\partial V}{\partial h}\ \mathrm{d}h \\
&= \frac{2}{3}\pi r h\ \mathrm{d}r + \frac{\pi}{3}r^2\ \mathrm{d}h \\
&= \frac{2}{3}\pi(300)(\pm\frac{1}{10}) - \frac{\pi}{3}(100)(\pm\frac{6}{10})
\end{aligned}$$

At the ends of the ranges, the maximum change in volume based on the error is $20\pi + 20\pi = 40\pi$.

Practice Problem

Find $\frac{\mathrm{d}z}{\mathrm{d}x}$ given the curve:

$$yz + x\ln(y) - z^2 = 0$$

$$\frac{\mathrm{d}z}{\mathrm{d}x} = -\frac{F_x}{F_z} = -\frac{\ln(y)}{y - 2z}$$

Practice Problem

Find the line tangent to the curve $C = S_1 \cap S_2$ at the point (3,4,5):

$$S_1 : z = \sqrt{x^2 + y^2}$$

$$S_2 : z = 1 + y$$

$$\begin{aligned}
\vec{u} &= \overrightarrow{\nabla F_1} \times \overrightarrow{\nabla F_1} \\
&= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{-x}{\sqrt{x^2+y^2}} & \frac{-y}{\sqrt{x^2+y^2}} & 1 \\ 0 & -1 & 1 \end{vmatrix} \\
&= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\frac{3}{5} & -\frac{4}{5} & 1 \\ 0 & -1 & 1 \end{vmatrix} \\
&= \langle \frac{1}{5}, \frac{3}{5}, \frac{3}{5} \rangle \parallel \langle 1, 3, 3 \rangle
\end{aligned}$$

$$l_{tangent} = \begin{cases} x &= 3 + t \\ y &= 4 + 3t \\ z &= 5 + 3t \end{cases}$$

Practice Problem

Evaluate the directional derivative when the vector $\vec{u}$ goes from (2,1) to (5,5).

$$D_{\vec{u}}(x^2 + xy^3)(2, 1)$$

$$
\begin{aligned}
D_{\vec{u}}(x^2 + xy^3)(2,1) &= \overrightarrow{\nabla f} \cdot \vec{u} \\
&= \langle 2x + y^3, 2xy^2 \rangle \cdot \langle \frac{3}{5}, \frac{4}{5} \rangle \\
&= \langle 5, 6 \rangle \cdot \langle \frac{3}{5}, \frac{4}{5} \rangle \\
&= \frac{39}{5}
\end{aligned}
$$

Practice Problem

Approximate the value of $(0.9)^2(3.9)^2$:
We can take (1,4) as a reference point using the curve x^2y^2.

$$
\begin{aligned}
x^2y^2 &\approx 16 + (2xy^2)(x-1) + (2x^y)(y-1) \\
&\approx 16 + 32(x-1) + 8(y-4) \\
(0.9)^2(3.9)^2 &\approx 16 + 32(-0.1) + 8(-0.1) \\
&\approx 16 - 3.2 - 0.8 \\
&\approx 12
\end{aligned}
$$

You can find all my notes at `http://omgimanerd.tech/notes`. If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech