Multivariable and Vector Calculus: Homework 12

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Section 16.9

Exercise 7

Use the Divergence Theorem to calculate the surface integral $\iint_S F \cdot \, \mathrm{d}S$; that is, calculate the flux of F across S.

$$F(x, y, z) = 3xy^2\hat{\imath} + xe^z\hat{\jmath} + z^3\hat{k}$$

S is the surface of the solid bounded by the cylinder $y^2 + z^2 = 1$ and the planes $x = -1$ and $x = 2$.

$$\begin{aligned}
\iint_S F \, \mathrm{d}S &= \iiint_E \text{div } F \, \mathrm{d}V \\
\text{div } F &= 3y^2 + 0 + 3z^2 \\
3\iiint_E y^2 + z^2 \, \mathrm{d}V &= 3\int_{-1}^{2}\int_{0}^{2\pi}\int_{0}^{1} (r\cos\theta)^2 + (r\sin\theta)^2 r \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}z \\
&= 3\int_{-1}^{2}\int_{0}^{2\pi}\int_{0}^{1} r^3(\cos^2\theta + \sin^2\theta) \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}z \\
&= 3\int_{-1}^{2}\int_{0}^{2\pi} \left[\frac{r^4}{4}\right]_0^1 \mathrm{d}\theta \, \mathrm{d}z \\
&= \frac{3}{4}\int_{-1}^{2}\int_{0}^{2\pi} \mathrm{d}\theta \, \mathrm{d}z \\
&= \frac{3}{4}\int_{-1}^{2} \left[\theta\right]_0^{2\pi} \mathrm{d}z \\
&= \frac{6\pi}{4}\int_{-1}^{2} \mathrm{d}z \\
&= \frac{3\pi}{2}\left[z\right]_{-1}^{2} \\
&= \frac{9\pi}{2}
\end{aligned}$$

Exercise 11

Use the Divergence Theorem to calculate the surface integral $\iint_S F \cdot \, \mathrm{d}S$; that is, calculate the flux of F across S.

$$F(x, y, z) = (2x^3 + y^3)\hat{\imath} + (y^3 + z^3)\hat{\jmath} + 3y^2z\hat{k}$$

S is the surface of the solid bounded by the paraboloid $z = 1 - x^2 - y^2$ and the xy-plane.

$$\begin{aligned}
\iint_S F \; \mathrm{d}S &= \iiint_E \text{div } F \; \mathrm{d}V \\
\text{div } F &= 6x^2 + 3y^2 + 3y^2 \\
&= 6(x^2 + y^2)
\end{aligned}$$

$$\begin{aligned}
\iiint_E 6(x^2+y^2) \; \mathrm{d}V &= 6 \int_0^{2\pi} \int_0^1 \int_0^{1-r^2} r^2 r \; \mathrm{d}z \; \mathrm{d}r \; \mathrm{d}\theta \\
&= 6 \int_0^{2\pi} \int_0^1 r^3 \Big[z \Big]_0^{1-r^2} \mathrm{d}r \; \mathrm{d}\theta \\
&= 6 \int_0^{2\pi} \int_0^1 r^3 (1 - r^2) \; \mathrm{d}r \; \mathrm{d}\theta \\
&= 6 \int_0^{2\pi} \left[\frac{r^4}{4} - \frac{r^6}{6} \right]_0^1 \mathrm{d}\theta \\
&= 6 \int_0^{2\pi} \frac{1}{4} - \frac{1}{6} \; \mathrm{d}\theta \\
&= \frac{6}{12} \int_0^{2\pi} \mathrm{d}\theta \\
&= \frac{2\pi}{2} = \pi
\end{aligned}$$

Exercise 17

Use the Divergence Theorem to evaluate $\iint_S F \cdot \mathrm{d}S$, where $F(x, y, z) = z^2 x \hat{\imath} + (\frac{1}{3}y^3 + \tan(z))\hat{\jmath} + (x^2 z + y^2)\hat{k}$ and S is the top half of the sphere $x^2 + y^2 + z^2 = 1$. Note that S is not a closed surface. First compute integrals over S_1 and S_2, where S_1 is the disk $x^2 + y^2 \leq 1$, oriented downward, and $S_2 = S \cup S_1$.

$$\begin{aligned}
\iint_S F \; \mathrm{d}S &= \iiint_E \text{div } F \; \mathrm{d}V \\
\text{div } F &= z^2 + y^2 + x^2
\end{aligned}$$

$$\begin{aligned}
\iint_{S_1} F \; \mathrm{d}S &= \iint_D F \cdot (-\hat{k}) \sqrt{(\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2 + 1} \; \mathrm{d}A \\
&= -\iint_D (x^2 z + y^2)\sqrt{1} \; \mathrm{d}A \\
&= -\int_0^{2\pi} \int_0^1 (r\cos\theta)^2(0) + (r\sin\theta)^2 r \; \mathrm{d}r \; \mathrm{d}\theta \\
&= -\int_0^{2\pi} \sin^2\theta \int_0^1 r^3 \; \mathrm{d}r \; \mathrm{d}\theta \\
&= -\int_0^{2\pi} \sin^2\theta \left[\frac{r^4}{4} \right]_0^1 \mathrm{d}\theta
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{4}\int_0^{2\pi} \frac{1-\cos(2\theta)}{2}\ \mathrm{d}\theta \\
&= -\frac{1}{8}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{2\pi} \\
&= -\frac{2\pi}{8} = -\frac{\pi}{4}
\end{aligned}
$$

Exercise 25

Prove each identity, assuming that S and E satisfy the conditions of the Divergence Theorem and the scalar functions and components of the vector fields have continouous second-order partial derivatives.

$$\iint_S a \cdot n\ \mathrm{d}S = 0$$

where $\vec{a}$ is a constant vector.

$$
\begin{gathered}
\text{div } \vec{a} = 0 \\
\iint_S \vec{a} \cdot n\ \mathrm{d}S = \iiint_E 0\ \mathrm{d}V = 0
\end{gathered}
$$

Exercise 27

Prove each identity, assuming that S and E satisfy the conditions of the Divergence Theorem and the scalar functions and components of the vector fields have continouous second-order partial derivatives.

$$\iint_S \text{curl } F \cdot\ \mathrm{d}S = 0$$

$$
\begin{aligned}
\iint_S \text{curl } F \cdot\ \mathrm{d}S &= \iiint_E \text{div}(\text{curl } F)\ \mathrm{d}V \\
&= \iint_E \nabla \cdot (\nabla \times F)\ \mathrm{d}V \\
&= \iiint_E 0\ \mathrm{d}V \\
&= 0
\end{aligned}
$$

Section 16.8

Exercise 5

Use Stokes' Theorem to evaluate $\iint_S \text{curl } F \cdot\ \mathrm{d}S$.

$$F(x,y,z) = xyz\hat{\imath} + xy\hat{\jmath} + x^2yz\hat{k}$$

S consists of the top and the four sides (but not the bottom) of the cube with vertices $(\pm1, \pm1, \pm1)$, oriented outward.

$$\begin{aligned}
\iint_S \text{curl}\ \ \mathrm{d}S &= \iint_H \text{curl}\ F\hat{k}\ \mathrm{d}S \\
&= \iint_H \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\ \mathrm{d}S \\
&= \iint_H y - zx\ \mathrm{d}S \\
&= \iint_S y + x\ \mathrm{d}S \\
&= \int_{-1}^{1}\int_{-1}^{1} y + x\ \mathrm{d}x\ \mathrm{d}y \\
&= \int_{-1}^{1} \left[yx + \frac{x^2}{2}\right]_{-1}^{1}\ \mathrm{d}y \\
&= \int_{-1}^{1} 2y\ \mathrm{d}y \\
&= 0
\end{aligned}$$

Exercise 9

Use Stokes' Theorem to evaluate $\int_C F\cdot\ \mathrm{d}r$. In each case C is oriented counterclockwise as viewed from above.

$$F(x, y, z) = xy\hat{\imath} + yz\hat{\jmath} + zx\hat{k}$$

C is the boundary of the part of the paraboloid $1 - x^2 - y^2$ in the first octant.

$$\begin{aligned}
\int_C F\cdot\ \mathrm{d}r &= \iint_S \text{curl}\ F\ \mathrm{d}S \\
&= \iint_D -P(\frac{\partial g}{\partial x}) - Q(\frac{\partial g}{\partial y}) + R\ \mathrm{d}A \\
&= \iint_D y(-2x) - (-z)(-2y) - x\ \mathrm{d}A \\
&= \iint_D -2xy - 2yz - x\ \mathrm{d}A \\
&= \iint_D -2xy - 2y(1 - x^2 - y^2) - x\ \mathrm{d}A \\
&= \iint_D -2xy - 2y + 2yx^2 + 2y^3 - x\ \mathrm{d}A \\
&= \iint_D -2y(x + 1 - x^2 - y^2) - x\ \mathrm{d}A
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^{\frac{\pi}{2}} \int_0^1 (-2r\cos\theta(r\sin\theta + 1 - r^2) - r\cos\theta) r \ \mathrm{d}r \ \mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{2}} \int_0^1 -2r^3\cos\theta\sin\theta - 2r\cos\theta + 2r^4\cos\theta - r^2\cos\theta \ \mathrm{d}r \ \mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{2}} \int_0^1 -r^3\sin 2\theta - 2r\cos\theta + 2r^4\cos\theta - r^2\cos\theta \ \mathrm{d}r \ \mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{2}} \left[\frac{-r^4\sin 2\theta}{4} - \frac{2r^3\sin\theta}{3} + \frac{2r^5\sin\theta}{5} - \frac{r^3\cos\theta}{3}\right]_0^1 \mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{2}} -\frac{\sin 2\theta}{4} - \frac{2\sin\theta}{3} + \frac{2\sin\theta}{5} - \frac{\cos\theta}{3} \ \mathrm{d}\theta \\
&= \left[\frac{1}{2}\frac{\cos 2\theta}{4} + \frac{2\cos\theta}{3} - \frac{2\cos\theta}{5} - \frac{\sin\theta}{3}\right]_0^{\frac{\pi}{2}} \\
&= (-\frac{1}{8} + 0 - 0 - \frac{1}{3}) - (\frac{1}{8} + \frac{2}{3} - \frac{2}{5} - 0) \\
&= -\frac{17}{20}
\end{aligned}
$$

Exercise 15

Verify that Stokes' Theorem is true for the given vector field F and surface S.

$$F(x, y, z) = y\hat{i} + z\hat{j} + x\hat{k}$$

S is the hemisphere $x^2 + y^2 + z^2 = 1$, $y \geq 0$, oriented in the direction of the positive y-axis.

?

Exercise 17

A particle moves along line segments from the origin to the points (1,0,0), (1,2,1), (0,2,1), and back to the origin under the influence of the force field

$$F(x, y, z) = z^2\hat{i} + 2xy\hat{j} + 4y^2\hat{k}$$

Find the work done.

$$\begin{aligned}
\int_C F\cdot\ \mathrm{d}r &= \iint_S \text{curl } F\ \mathrm{d}S \\
&= \iint_D -P(\frac{\partial g}{\partial x}) - Q(\frac{\partial g}{\partial y}) + R\ \mathrm{d}A \\
&= \iint_D -8y(\frac{\partial g}{\partial x}) - 2z(\frac{\partial g}{\partial y}) + 2y\ \mathrm{d}A \\
&= \iint_D 0 - z + 2y\ \mathrm{d}A \\
&= \iint_D 0 - \frac{y}{2} + 2y\ \mathrm{d}A \\
&= \iint_D \frac{3y}{2}\ \mathrm{d}A \\
&= \frac{3}{2}\int_0^1\int_0^2 y\ \mathrm{d}y\ \mathrm{d}x \\
&= \frac{3}{2}\int_0^1 2\ \mathrm{d}x \\
&= 3
\end{aligned}$$

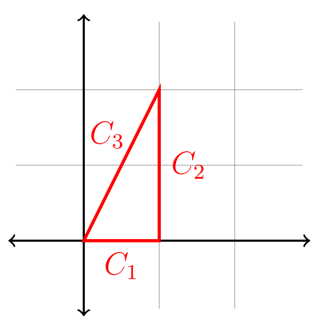
Section 16.4

Exercise 3

Evaluate the line integral directly and using Green's Theorem.

$$\oint_C xy\ \mathrm{d}x + x^2y^3\ \mathrm{d}y$$

C is the triangle with vertices (0,0), (1,0), and (1,2)



$$
\begin{aligned}
\oint_C xy \, \mathrm{d}x + x^2y^3 \, \mathrm{d}y &= \int_{C_1} xy \, \mathrm{d}x + x^2y^3 \, \mathrm{d}y + \int_{C_2} xy \, \mathrm{d}x + x^2y^3 \, \mathrm{d}y + \int_{C_3} xy \, \mathrm{d}x + x^2y^3 \, \mathrm{d}y \\
&= \int_{C_1} 0 \, \mathrm{d}x + 0 + \int_{C_2} 0 + y^3 \, \mathrm{d}y + \int_{C_3} \frac{y}{2} y \frac{\mathrm{d}y}{2} + (\frac{y}{2})^2 y^3 \, \mathrm{d}y \\
&= 0 + \int_0^2 y^3 \, \mathrm{d}y + \int_2^0 \frac{y^2}{4} \, \mathrm{d}y + \frac{y^5}{4} \, \mathrm{d}y \\
&= 0 + \left[\frac{y^4}{4}\right]_0^2 + \left[\frac{y^3}{12} + \frac{y^6}{24}\right]_2^0 \\
&= 4 - \frac{2}{3} - \frac{8}{3} \\
&= \frac{2}{3}
\end{aligned}
$$

$$
\begin{aligned}
\oint_C xy \, \mathrm{d}x + x^2y^3 \, \mathrm{d}y &= \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \, \mathrm{d}A \\
&= \int_0^1 \int_0^{2x} -x + 2xy^3 \, \mathrm{d}y \, \mathrm{d}x \\
&= \int_0^1 \left[- xy + \frac{xy^4}{2}\right]_0^{2x} \, \mathrm{d}x \\
&= \int_0^1 -2x^2 + 8x^5 \, \mathrm{d}x \\
&= \left[- \frac{2x^3}{3} + \frac{8x^6}{6}\right]_0^1 \\
&= -\frac{2}{3} + \frac{8}{6} \\
&= \frac{2}{3}
\end{aligned}
$$

Exercise 7

Use Green's Theorem to evaluate the line integral along the given positively oriented curve.

$$\int_C (y + \mathrm{e}^{\sqrt{x}}) \, \mathrm{d}x + (2x + \cos(y^2)) \, \mathrm{d}y$$

C is the boundary of the region enclosed by the parabolas $y = x^2$ and $x = y^2$.

$$
\begin{aligned}
\oint_C P \, \mathrm{d}x + Q \, \mathrm{d}y &= \iint_D \left(\frac{\partial Q}{\partial x} + \frac{\partial P}{\partial y}\right) \, \mathrm{d}A \\
&= \int_0^1 \int_{x^2}^{\sqrt{x}} (2 - 1) \, \mathrm{d}y \, \mathrm{d}x \\
&= \int_0^1 \left[y\right]_{x^2}^{\sqrt{x}} \, \mathrm{d}x \\
&= \int_0^1 \sqrt{x} - x^2 \, \mathrm{d}x \\
&= \left[\frac{2x^{\frac{3}{2}}}{3} - \frac{x^3}{3}\right]_0^1 \\
&= \frac{2}{3} - \frac{1}{3} = \frac{1}{3}
\end{aligned}
$$

Exercise 13

Use Green's Theorem to evaluate $\int_C F \cdot \, \mathrm{d}r$.

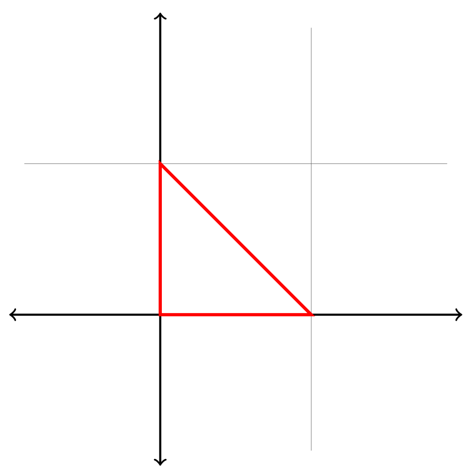
$$F(x, y) = \langle y - \cos(y), x\sin(y) \rangle$$

C is the circle $(x-3)^2 + (y+4)^2 = 4$ oriented clockwise.

$$\begin{aligned}
\oint_C P \, \mathrm{d}x + Q \, \mathrm{d}y &= \iint_D \left(\frac{\partial Q}{\partial x} + \frac{\partial P}{\partial y} \right) \, \mathrm{d}A \\
&= \iint_D \sin(y) + 1 - \sin(y) \, \mathrm{d}x \, \mathrm{d}y \\
&= \int_0^2 \int_0^{2\pi} r \, \mathrm{d}\theta \, \mathrm{d}r \\
&= \int_0^2 \Big[r\theta \Big]_0^{2\pi} \, \mathrm{d}r \\
&= \int_0^2 2\pi r \, \mathrm{d}r \\
&= \Big[\pi r^2 \Big]_0^2 \\
&= 4\pi
\end{aligned}$$

Exercise 17

Use Green's Theorem to find the work done by the force $F(x, y) = x(x+y)\hat{\imath} + xy^2\hat{\jmath}$ in moving a particle from the origin along the x-axis to (1,0), then along the line segment to (0,1), and then back to the origin along the y-axis.



$$
\begin{aligned}
\oint_C P\ \mathrm{d}x + Q\ \mathrm{d}y &= \iint_D \left(\frac{\partial Q}{\partial x} + \frac{\partial P}{\partial y}\right)\ \mathrm{d}A \\
&= \iint_D y^2 - x\ \mathrm{d}A \\
&= \int_0^1 \int_0^{1-x} y^2 - x\ \mathrm{d}y\ \mathrm{d}x \\
&= \int_0^1 \left[\frac{y^3}{3} - xy\right]_0^{1-x}\ \mathrm{d}x \\
&= \int_0^1 \frac{(1-x)^3}{3} - x(1-x)\ \mathrm{d}x \\
&= \int_0^1 \frac{1-3x+3x^2-x^3}{3} - x + x^2\ \mathrm{d}x \\
&= \int_0^1 \frac{1}{3} - x + x^2 - \frac{x^3}{3} - x + x^2\ \mathrm{d}x \\
&= \int_0^1 -\frac{x^3}{3} + 2x^2 - 2x + \frac{1}{3}\ \mathrm{d}x \\
&= \left[-\frac{x^4}{12} + \frac{2x^3}{3} - x^2 + \frac{x}{3}\right]_0^1 \\
&= -\frac{1}{12} + \frac{2}{3} - 1 + \frac{1}{3} \\
&= -\frac{1}{12}
\end{aligned}
$$

If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech