Multivariable and Vector Calculus: Homework 10

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Section 16.2

Exercise 3

Evaluate the line integral, where C is the given curve.

$$\int_C xy^4 \ \mathrm{d}s \quad C \text{ is the right half of the circle } x^2 + y^2 = 16$$

$$\begin{aligned}
\mathrm{d}s &= \sqrt{(-4\sin(t))^2 + (4\cos(t))^2} \ \mathrm{d}t \\
&= 4 \ \mathrm{d}t \\
\int_C xy^4 \ \mathrm{d}s &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (4\cos(t))(4\sin(t))^4(4 \ \mathrm{d}t) \\
&= 4^6 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(t)\sin^4(t) \ \mathrm{d}t \\
&= 4^6 \left[\frac{\sin^5(t)}{5}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\
&= \frac{4096(2)}{5} = \frac{8192}{5}
\end{aligned}$$

Exercise 11

Evaluate the line integral, where C is the given curve.

$$\int_C x\mathrm{e}^{yz} \ \mathrm{d}s \quad C \text{ is the line segment from } (0,0,0) \text{ to } (1,2,3)$$

$$\begin{aligned}
c(t) &= \langle t, 2t, 3t\rangle \quad 0 \le t \le 1 \\
c'(t) &= \langle 1, 2, 3\rangle \\
|c'(t)| &= \sqrt{14} \\
\mathrm{d}s &= \sqrt{14}\ \mathrm{d}t \\
\int_C x\mathrm{e}^{yz}\ \mathrm{d}s &= \int_0^1 t\mathrm{e}^{(2t)(3t)}\sqrt{14}\ \mathrm{d}t \\
&= \sqrt{14}\int_0^1 t\mathrm{e}^{6t^2}\ \mathrm{d}t \\
&= \sqrt{14}\left[\frac{\mathrm{e}^{6x^2}}{12}\right]_0^1 \\
&= \sqrt{14}\frac{\mathrm{e}^6 - 1}{12}
\end{aligned}$$

Exercise 21

Evaluate the line integral $\int_C F\ \mathrm{d}r$, where C is given by the vector function $r(t)$.

$$F(x,y,z) = \sin(x)\hat{\imath} + \cos(y)\hat{\jmath} + xz\hat{k} \quad r(t) = t^3\hat{\imath} - t^2\hat{\jmath} + t\hat{k} \quad (0 \le t \le 1)$$

$$\begin{aligned}
r(t) &= t^3\hat{\imath} - t^2\hat{\jmath} + t\hat{k} \\
r'(t) &= 3t^2\hat{\imath} - 2t\hat{\jmath} + \hat{k} \\
\mathrm{d}r &= (3t^2\hat{\imath} - 2t\hat{\jmath} + \hat{k})\ \mathrm{d}t \\
F(x,y,z) &= \sin(x)\hat{\imath} + \cos(y)\hat{\jmath} + xz\hat{k} \\
&= \sin(t^3)\hat{\imath} + \cos(-t^2)\hat{\jmath} + t^3(t)\hat{k} \\
\int_C F\cdot\ \mathrm{d}r &= \int_0^1 3t^2\sin(t^3) - 2t\cos(t^2) + t^4\ \mathrm{d}t \\
&= \left[-\cos(t^3) - \sin(t^2) + \frac{t^5}{5}\right]_0^1 \\
&= \frac{6}{5} - \cos(1) - \sin(1)
\end{aligned}$$

Exercise 25

Use a calculator to evaluate the line integral correct to four decimal places.

$$\int_C xy\tan^{-1} z\ \mathrm{d}s \quad \text{where } C \text{ has parametric equations } x = t^2, y = t^3, z = \sqrt{t}, 1 \le t \le 2$$

$$\begin{aligned}
r(t) &= \langle t^2, t^3, \sqrt{t}\rangle \\
r'(t) &= \langle 2t, 3t^2, \frac{1}{2\sqrt{t}}\rangle \\
|r'(t)| &= \sqrt{4t^2 + 9t^2 + \frac{1}{4t}} \\
\int_C xy\tan^{-1} z\ \mathrm{d}s &= \int_1^2 (t^2)(t^3)\tan^{-1}\sqrt{t}\sqrt{4t^2 + 9t^2 + \frac{1}{4t}}\ \mathrm{d}t \\
&\approx 94.8232
\end{aligned}$$

Exercise 32

Find the work done by the force field $F(x, y) = x^2\hat{\imath} + xy\hat{\jmath}$ on a particle that moves once around the circle $x^2 + y^2 = 4$ oriented in the counter clockwise direction.

$$\begin{aligned}
r(t) &= 2\cos(t)\hat{\imath} + 2\sin(t)\hat{\jmath} \quad (0 \leq t \leq 2\pi) \\
r'(t) &= -2\sin(t)\hat{\imath} + 2\cos(t)\hat{\jmath} \\
\int_C F \cdot \, \mathrm{d}r &= \int_0^{2\pi} 4\cos^2(t)(-2\sin(t)) + (2\cos(t))(2\sin(t))(2\cos(t)) \, \mathrm{d}t \\
&= \int_0^{2\pi} -8\cos^2(t)\sin(t) + 8\cos^2(t)\sin(t) \, \mathrm{d}t \\
&= \int_0^{2\pi} 0 \, \mathrm{d}t \\
&= 0
\end{aligned}$$

Exercise 41

Find the work done by the force field

$$F(x, y, z) = \langle x - y^2, y - z^2, z - x^2 \rangle$$

on a particle that moves along the line segment from (0,0,1) to (2,1,0).

$$\begin{aligned}
r(t) &= \langle 2t, t, 1 - t \rangle \quad (0 \leq t \leq 1) \\
r'(t) &= \langle 2, 1, -1 \rangle \\
F(x, y, z) &= \langle 2t - t^2, -t^2 + 3t - 1, 1 - t - 4t^2 \rangle \\
\int_0^1 F(x, y, z) \cdot r'(t) \, \mathrm{d}t &= \int_0^1 2(2t - t^2) + (-t^2 - 3t - 1) - (1 - t - 4t^2) \, \mathrm{d}t \\
&= \int_0^1 4t - 2t^2 - t^2 + 3t - 1 - 1 + t + 4t^2 \, \mathrm{d}t \\
&= \int_0^1 t^2 + 8t - 2 \, \mathrm{d}t \\
&= \left[\frac{t^3}{3} + \frac{8t^2}{2} - 2t \right]_0^1 \\
&= \frac{1}{3} + 4 - 2 \\
&= \frac{7}{3}
\end{aligned}$$

If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech