Multivariable and Vector Calculus: Homework 8

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Section 15.3

Exercise 9

Evaluate the given integral by changing to polar coordinates.

$$\iint_R \sin(x^2+y^2)\ \mathrm{d}A$$

where R is the region in the first quadrant between the circles with center the origin and radii 1 and 3.

$$\begin{aligned}
\iint_R \sin(x^2+y^2)\ \mathrm{d}A &= \int_0^{\frac{\pi}{2}}\int_1^3 \sin(r^2)r\ \mathrm{d}r\ \mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{2}} \left[-\frac{\cos(r^2)}{2}\right]_1^3 \mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{2}} \frac{1}{2}(-\cos(9)+\cos(1))\ \mathrm{d}\theta \\
&= \frac{1}{2}\left[\cos(1)\theta - \cos(9)\theta\right]_0^{\frac{\pi}{2}} \\
&= \frac{\pi(\cos(1)-\cos(9))}{4} \\
&= \frac{\pi}{4}(\cos(1)-\cos(9))
\end{aligned}$$

Exercise 15

Use a double integral to find the area of one loop of the rose $r=\cos(3\theta)$.

$$\begin{aligned}
\iint_D \mathrm{d}A &= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\int_0^{\cos(3\theta)} r\ \mathrm{d}r\ \mathrm{d}\theta \\
&= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left[\frac{r^2}{2}\right]_0^{\cos(3\theta)} \mathrm{d}\theta \\
&= \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos^2(3\theta)\ \mathrm{d}\theta
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{(1+\cos(6\theta))}{2}\ \mathrm{d}\theta \\
&= \frac{1}{4}\left[\theta - \frac{\sin(6\theta)}{6}\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \\
&= \frac{1}{4}\left(\frac{\pi}{4} - \frac{1}{6}\sin(\frac{6\pi}{4}) + \frac{\pi}{4} + \frac{1}{6}\sin(-\frac{6\pi}{4})\right) \\
&= \frac{\pi}{8}
\end{aligned}
$$

Exercise 17

Use a double integral to find the area of the region inside the circle $(x-1)^2+y^2=1$ and outside the circle $x^2+y^2=1$.

$$
\begin{aligned}
(x-1)^2+y^2 &= 1 \\
x^2-2x+1+y^2 &= 1 \\
x^2+y^2-2x &= 0 \\
r^2-2r\cos\theta &= 0 \\
r &= 2\cos\theta \\
x^2+y^2 &= 1 \\
r &= 1 \\
r &= 1 = 2\cos\theta \\
\theta &= -\frac{\pi}{3} \quad \theta = \frac{\pi}{3} \\
\iint_D \mathrm{d}A &= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}\int_{1}^{2\cos\theta} r\ \mathrm{d}r\ \mathrm{d}\theta \\
&= 2\int_0^{\frac{\pi}{3}} \left[\frac{r^2}{2}\right]_1^{2\cos\theta} \mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{3}} 4\cos^2\theta - 1\ \mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{3}} 4\frac{\cos 2\theta + 1}{2} - 1\ \mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{3}} 2\cos 2\theta - 1\ \mathrm{d}\theta \\
&= \Big[-\sin(2\theta) - \theta\Big]_0^{\frac{\pi}{3}} \\
&= \sin(\frac{2\pi}{3} - \frac{\pi}{3} + \sin(0)) \\
&= \frac{\sqrt{3}}{2} - \frac{\pi}{3}
\end{aligned}
$$

Exercise 21

Use polar coordinates to find the volume below the plane $2x + y + z = 4$ and above the disk $x^2 + y^2 \leq 1$.

$$\begin{aligned}
x(x, y) &= 4 - 2x - y \\
\iint_D 4 - 2x - y \, \mathrm{d}A &= \int_0^{2\pi} \int_0^1 4 - 2(r\cos\theta) - (r\sin\theta) r \, \mathrm{d}r \, \mathrm{d}\theta \\
&= \int_0^{2\pi} \left[2r - \frac{2r^2 \sin\theta}{3} + \frac{r^2 \cos\theta}{3} \right]_0^1 \mathrm{d}\theta \\
&= \int_0^{2\pi} 2 - \frac{2}{3}\sin\theta + \frac{1}{3}\cos\theta \, \mathrm{d}\theta \\
&= \left[2\theta + \frac{2}{3}\cos\theta - \frac{1}{3}\sin\theta \right]_0^{2\pi} \\
&= 4\pi + \frac{2}{3}\cos 2\pi - \frac{1}{2}\sin 2\pi - 0 - \frac{2}{3}\cos 0 + \frac{1}{3}\sin 0 \\
&= 4\pi
\end{aligned}$$

Exercise 29

Evaluate the iterated integral by converting it to polar coordinates.

$$\begin{aligned}
\int_0^2 \int_0^{\sqrt{4-x^2}} \mathrm{e}^{-x^2-y^2} \, \mathrm{d}y \, \mathrm{d}x &= \int_0^{\frac{\pi}{2}} \int_0^2 \mathrm{e}^{(-1)r^2} r \, \mathrm{d}r \, \mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{2}} \left[-\frac{\mathrm{e}^{-r^2}}{2} \right]_0^2 \mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{2}} \frac{1}{2} - \frac{\mathrm{e}^{-4}}{2} \, \mathrm{d}\theta \\
&= \frac{1}{2} \left[\theta - \frac{\theta}{\mathrm{e}^4} \right]_0^{\frac{\pi}{2}} \\
&= \frac{\pi}{4} - \frac{\pi}{4\mathrm{e}^4}
\end{aligned}$$

Exercise 31

Evaluate the iterated integral by converting it to polar coordinates.

$$\begin{aligned}
\int_0^{\frac{1}{2}} \int_{\sqrt{3}y}^{\sqrt{1-y^2}} xy^2 \, \mathrm{d}x \, \mathrm{d}y &= \int_0^{\frac{\pi}{6}} \int_0^1 (r\cos\theta)(r\sin\theta)^2 r \, \mathrm{d}r \, \mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{6}} \int_0^1 r^4 \cos\theta \sin^2\theta \, \mathrm{d}r \, \mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{6}} \left[\frac{r^5 \cos\theta \sin^2\theta}{5}\right]_0^1 \mathrm{d}\theta \\
&= \frac{1}{5}\int_0^{\frac{\pi}{6}} \cos\theta \sin^2\theta \, \mathrm{d}\theta \\
&= \frac{1}{5}\left[\frac{\sin^3\theta}{3}\right]_0^{\frac{\pi}{6}} \\
&= \frac{1}{15}\left(\sin^3(\frac{\pi}{6}) - \sin^3(0)\right) \\
&= \frac{1}{15}(\frac{1^3}{2^3} - 0) \\
&= \frac{1}{120}
\end{aligned}$$

Section 15.4

Exercise 7

Find the mass and center of mass of the lamina that occupies the region D and has the given density function ρ.

$$D \text{ is bounded by } y = 1 - x^2 \text{ and } y = 0; \quad \rho(x, y) = ky$$

$$\begin{aligned}
0 &= 1 - x^2 \\
x &= \pm 1 \\
mass &= \iint_D \rho(x, y) \, \mathrm{d}A \\
&= \int_{-1}^1 \int_0^{1-x^2} ky \, \mathrm{d}y \, \mathrm{d}x \\
&= 2\int_0^1 \left[\frac{ky^2}{2}\right]_0^{1-x^2} \mathrm{d}x \\
&= k\int_0^1 1 - 2x^2 + x^4 \, \mathrm{d}x \\
&= k\left[x - \frac{2x^3}{3} + \frac{x^5}{5}\right]_0^1 \\
&= k(1 - \frac{2}{3} + \frac{1}{5}) \\
&= \frac{8k}{15}
\end{aligned}$$

$$\begin{aligned}
x_{CoM} &= \frac{1}{mass}\iint_D x\rho(x,y)\ \mathrm{d}A \\
&= \frac{15}{8k}\int_{-1}^{1}\int_{0}^{1-x^2} xky\ \mathrm{d}y\ \mathrm{d}x \\
&= k\frac{15}{8k}\int_{-1}^{1}\int_{0}^{1-x^2} xy\ \mathrm{d}y\ \mathrm{d}x \\
&= \frac{15}{8}\int_{-1}^{1}\left[\frac{xy^2}{2}\right]_0^{1-x^2}\ \mathrm{d}x \\
&= \frac{15}{16}\int_{-1}^{1} x(1-2x^2+x^4)\ \mathrm{d}x \\
&= \frac{15}{16}\int_{-1}^{1} x-2x^3+x^5\ \mathrm{d}x \\
&= \frac{15}{16}\left[\frac{x^2}{2}-\frac{2x^4}{4}+\frac{x^6}{6}\right]_{-1}^{1} \\
&= 0
\end{aligned}$$

$$\begin{aligned}
y_{CoM} &= \frac{1}{mass}\iint_D y\rho(x,y)\ \mathrm{d}A \\
&= \frac{15}{8k}\int_{-1}^{1}\int_{0}^{1-x^2} ky^2\ \mathrm{d}y\ \mathrm{d}x \\
&= \frac{15}{8}\int_{-1}^{1}\left[\frac{y^3}{3}\right]_0^{1-x^2}\ \mathrm{d}y \\
&= \frac{5}{8}\int_{-1}^{1}(1-x^2)^3\ \mathrm{d}x \\
&= \frac{5}{8}\int_{-1}^{1} 1-x^6-3x^2+3x^4\ \mathrm{d}x \\
&= \frac{5}{8}\left[x-\frac{x^7}{7}-x^3+\frac{3x^5}{5}\right]_{-1}^{1} \\
&= \frac{5}{8}(1-\frac{1}{7}-1+\frac{3}{5}+1-\frac{1}{7}-1+\frac{3}{5}) \\
&= \frac{5}{8}(-\frac{2}{7}+\frac{6}{5}) \\
&= \frac{5}{8}\frac{32}{35} \\
&= \frac{4}{7}
\end{aligned}$$

Center of mass: $\langle 0, \frac{4}{7}\rangle$

Exercise 15

Find the center of mass of a lamina in the shape of an isosceles right triangle with equal sides of length a is the density at any point is proportional to the square of the distance from the vertex opposite the hypotenuse.
Treat the origin as the right angle with the legs a extending along the x and y axes.

$$\rho(x,y) = x^2+y^2$$

$$
\begin{aligned}
mass &= \iint_D \rho(x,y)\ \mathrm{d}A \\
&= \int_0^a \int_0^{a-y} x^2 + y^2\ \mathrm{d}x\ \mathrm{d}y \\
&= \int_0^a \left[\frac{x^3}{3} + xy^2\right]_0^{a-y}\ \mathrm{d}y \\
&= \int_0^a \frac{(a-y)^3}{3} + y^2(a-y)\ \mathrm{d}y \\
&= \int_0^a \frac{1}{3}(a^3 - 3a^2y + 3ay^2 - y^3) + \frac{1}{3}(3ay^2 - 3y^3)\ \mathrm{d}y \\
&= \frac{1}{3}\int_0^a a^3 - 3a^2y + 6ay^2 - 4y^3\ \mathrm{d}y \\
&= \frac{1}{3}\left[a^3y - \frac{3a^2y^2}{2} + 2ay^3 - y^4\right]_0^a \\
&= \frac{1}{3}(a^4 - \frac{3a^4}{2} + 2a^4 - a^4) \\
&= \frac{a^4}{6} \\
x_{CoM} = y_{CoM} &= \frac{1}{mass}\iint_D y\rho(x,y)\ \mathrm{d}A \\
&= \frac{6}{a^4}\int_0^a \int_0^{a-y} y(x^2+y^2)\ \mathrm{d}x\ \mathrm{d}y \\
&= \frac{6}{a^4}\int_0^a \left[\frac{x^3y}{3} + xy^3\right]_0^{a-y}\ \mathrm{d}y \\
&= \frac{6}{a^4}\int_0^a \frac{y}{3}(a^3 - 3a^2y + 3ay^2 - y^3) + y^3(a-y)\ \mathrm{d}y \\
&= \frac{6}{a^4}\int_0^a \frac{a^3y}{3} - a^2y^2 + ay^3 - \frac{y^4}{3} + ay^3 - y^4\ \mathrm{d}y \\
&= \frac{6}{a^4}\int_0^a \frac{a^3y}{3} - a^2y^2 + 2ay^3 - \frac{4y^4}{3}\ \mathrm{d}y \\
&= \frac{6}{a^4}\left[\frac{a^3y^2}{6} - \frac{a^2y^3}{3} + \frac{2ay^4}{4} - \frac{4y^5}{15}\right]_0^a \\
&= \frac{6}{a^4}\left[\frac{a^5}{6} - \frac{a^5}{3} + \frac{a^5}{2} - \frac{4a^5}{15}\right] \\
&= a - 2a + 3a - \frac{8a}{5} \\
&= \frac{2a}{5}
\end{aligned}
$$

Center of mass: $\langle \frac{2a}{5}, \frac{2a}{5} \rangle$

Section 15.6

Exercise 11

Evaluate the triple integral.

$$\begin{aligned}
\iiint_E \frac{z}{x^2+z^2}\ \mathrm{d}V &= \int_1^4 \int_y^4 \int_0^z \frac{z}{x^2+z^2}\ \mathrm{d}x\ \mathrm{d}z\ \mathrm{d}y \\
&= \int_1^4 \int_y^4 z\left[\frac{1}{z}\tan^{-1}(\frac{x}{z})\right]_0^z \mathrm{d}z\ \mathrm{d}y \\
&= \int_1^4 \int_y^4 \frac{\pi}{4}\ \mathrm{d}z\ \mathrm{d}y \\
&= \int_1^4 \left[\frac{\pi z}{4}\right]_y^4 \mathrm{d}y \\
&= \int_1^4 \pi - \frac{\pi y}{4}\ \mathrm{d}y \\
&= \left[\pi y - \frac{\pi y^2}{8}\right]_1^4 \\
&= 4\pi - 2\pi - (\pi - \frac{\pi}{8}) \\
&= \frac{9\pi}{8}
\end{aligned}$$

Exercise 13

Evaluate the triple integral $\iiint_E 6xy\ \mathrm{d}V$ where E lies below the plane $z = 1 + x + y$ and above the region in the xy-plane bounded by the curves $y = \sqrt{x}, y = 0, x = 1$.

$$\begin{aligned}
\iiint_E 6xy\ \mathrm{d}V &= \int_0^1 \int_0^{\sqrt{x}} \int_0^{1+x+y} 6xy\ \mathrm{d}z\ \mathrm{d}y\ \mathrm{d}x \\
&= \int_0^1 \int_0^{\sqrt{x}} \left[6xyz\right]_0^{1+x+y} \mathrm{d}y\ \mathrm{d}x \\
&= \int_0^1 \int_0^{\sqrt{x}} 6xy(1+x+y)\ \mathrm{d}y\ \mathrm{d}x \\
&= \int_0^1 \left[3xy^2 + 3x^2y^2 + 2xy^3\right]_0^{\sqrt{x}} \mathrm{d}x \\
&= \int_0^1 3x^2 + 3x^3 + 2x^{\frac{5}{2}}\ \mathrm{d}x \\
&= \left[x^3 + \frac{3x^4}{4} + \frac{4}{7}x^{\frac{4}{7}}\right]_0^1 \\
&= 1 + \frac{3}{4} + \frac{4}{7} \\
&= \frac{28}{28} + \frac{21}{28} + \frac{16}{28} \\
&= \frac{65}{28}
\end{aligned}$$

Exercise 21

Use a triple integral to find the volume of the solid enclosed by the cylinder $y = x^2$ and the planes $z = 0$ and $y + z = 1$.

$$
\begin{aligned}
&\{(x, y, z) \in E \mid -1 \leq x \leq 1, x^2 \leq y \leq 1, 0 \leq z \leq 1 - y\} \\
\iiint_E \mathrm{d}V &= \int_{-1}^{1} \int_{x^2}^{1} \int_{0}^{1-y} \mathrm{d}z \; \mathrm{d}y \; \mathrm{d}x \\
&= \int_{-1}^{1} \int_{x^2}^{1} \Big[z \Big]_0^{1-y} \mathrm{d}y \; \mathrm{d}x \\
&= \int_{-1}^{1} \int_{x^2}^{1} 1 - y \; \mathrm{d}y \; \mathrm{d}x \\
&= \int_{-1}^{1} \left[y - \frac{y^2}{2} \right]_{x^2}^{1} \mathrm{d}x \\
&= \int_{-1}^{1} 1 - \frac{1}{2} - x^2 + \frac{x^4}{2} \; \mathrm{d}x \\
&= \left[\frac{x^5}{10} - \frac{x^3}{3} + \frac{x}{2} \right]_{-1}^{1} \\
&= \frac{1}{10} - \frac{1}{3} + \frac{1}{2} + \frac{1}{10} - \frac{1}{3} + \frac{1}{2} \\
&= \frac{1}{5} - \frac{2}{3} + 1 \\
&= \frac{8}{15}
\end{aligned}
$$

Exercise 33

The figure shows the region of integration for the integral

$$\int_0^1 \int_{\sqrt{x}}^{1} \int_0^{1-y} f(x, y, z) \; \mathrm{d}z \; \mathrm{d}y \; \mathrm{d}x$$

Rewrite this integral as an equivalent iterated integral in the five other orders.

$$
\begin{aligned}
\int_0^1 \int_{\sqrt{x}}^{1} \int_0^{1-y} f(x, y, z) \; \mathrm{d}z \; \mathrm{d}y \; \mathrm{d}x &= \int_0^1 \int_0^{1-y} \int_0^{y^2} \mathrm{d}x \; \mathrm{d}y \; \mathrm{d}z \\
&= \int_0^1 \int_0^{1-z} \int_0^{y^2} \mathrm{d}x \; \mathrm{d}z \; \mathrm{d}y \\
&= \int_0^1 \int_0^{(1-z)^2} \int_{\sqrt{x}}^{1-z} \mathrm{d}y \; \mathrm{d}x \; \mathrm{d}z \\
&= \int_0^1 \int_0^{1-\sqrt{x}} \int_{\sqrt{x}}^{1-z} \mathrm{d}y \; \mathrm{d}z \; \mathrm{d}x \\
&= \int_0^1 \int_0^{y^2} \int_0^{1-y} \mathrm{d}z \; \mathrm{d}x \; \mathrm{d}y
\end{aligned}
$$

If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech