Multivariable and Vector Calculus: Homework 6

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Section 14.4

Exercise 5

Find an equation of the tangent plane to the given surface at the specified point.

$$z = x\sin(x+y) \quad (-1,1,0)$$

$$\begin{aligned}
z - z_0 &= f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) \\
\frac{\partial z}{\partial x} &= x\cos(x+y) + \sin(x+y) \\
\frac{\partial z}{\partial x}(-1,1) &= -1 + 0 = 0 \\
\frac{\partial z}{\partial y} &= x\cos(x+y) \\
\frac{\partial z}{\partial y}(-1,1) &= -1 \\
z &= -1(x+1) - 1(y-1) \\
-x - y &= z \\
x + y + z &= 0
\end{aligned}$$

Exercise 33

The length and width of a rectangle are measured as 30cm and 24cm, respectively, with an error in measurement of at most 0.1cm in each. Use differentials to estimate the maximum error in the calculated area of the rectangle.

$$\begin{aligned}
l &= 30 \\
w &= 24 \\
\mathrm{d}l &= 0.1 \\
\mathrm{d}w &= 0.1 \\
A &= lw \\
\mathrm{d}A &= \frac{\partial l}{\partial A}\ \mathrm{d}l + \frac{\partial w}{\partial A}\ \mathrm{d}w \\
&= w\ \ \mathrm{d}l + l\ \ \mathrm{d}w \\
&= 30(0.1) + 24(0.1) \\
&= 5.4cm^2
\end{aligned}$$

Exercise 35

Use differentials to estimate the amount of tin in a closed tin can with diameter 8cm and height 12cm if the tin is 0.04cm thick.

$$\begin{aligned}
d &= 8 \\
r &= 4 \\
h &= 12 \\
\mathrm{d}d &= \ \mathrm{d}r = 0.04 \\
\mathrm{d}h &= 0.08 \\
V &= \pi r^2 h \\
\mathrm{d}V &= \frac{\partial V}{\partial d}\ \mathrm{d}d + \frac{\partial V}{\partial h}\ \mathrm{d}h \\
&= 2\pi r h(0.04) + \pi r^2(0.08) \\
&= 16.0768 cm^2
\end{aligned}$$

Exercise 39

If R is the total resistance of three resistors, connected in parallel, with resistances R_1, R_2, R_3, then:

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

If the resistances are measured in ohms as $R_1 = 25\Omega, R_2 = 40\Omega, R_3 = 50\Omega$, with a possible error of 0.5% in each case, estimate the maximum error in the calculated value of R.

$$\begin{aligned}
R_1 &= 25 \quad \mathrm{d}R_1 = 0.125 \\
R_2 &= 40 \quad \mathrm{d}R_2 = 0.2 \\
R_3 &= 50 \quad \mathrm{d}R_3 = 0.25 \\
\frac{1}{R} &= \frac{1}{25} + \frac{1}{40} + \frac{1}{50} \\
R &= 11.764 \\
\frac{1}{R} &= \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} \\
\mathrm{d}R &= \frac{\partial R}{\partial R_1}\ \mathrm{d}R_1 + \frac{\partial R}{\partial R_2}\ \mathrm{d}R_2 + \frac{\partial R}{\partial R_3}\ \mathrm{d}R_3 \\
\frac{\partial R}{\partial R_1} &= \frac{R^2}{(R_1)^2} \\
\frac{\partial R}{\partial R_2} &= \frac{R^2}{(R_2)^2} \\
\frac{\partial R}{\partial R_3} &= \frac{R^2}{(R_3)^2} \\
\mathrm{d}R &= \frac{11.764^2}{25^2}(0.125) + \frac{11.764^2}{40^2}(0.2) + \frac{11.764^2}{50^2}(0.25) \\
&\approx 0.0588\Omega
\end{aligned}$$

Section 14.6

Exercise 5

Find the directional derivative of f at the given point in the direction indicated by the angle θ.

$$f(x,y) = y\cos(xy) \quad (0,1) \quad \theta = \frac{\pi}{4}$$

$$\begin{aligned}
\overrightarrow{\nabla F} &= \langle F_x, F_y \rangle \\
&= \big\langle -y^2 \sin(xy), -xy\sin(xy) + \cos(xy) \big\rangle \\
\overrightarrow{\nabla F}(0,1) &= \langle -1\sin(0), 0 + \cos(0) \rangle \\
&= \langle 0, 1 \rangle \\
D_{\vec{u}} f &= \overrightarrow{\nabla F}(0,1) \cdot \langle \cos\theta, \sin\theta \rangle \\
&= \langle 0, 1 \rangle \cdot \langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle \\
&= 0 + \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}
\end{aligned}$$

Exercise 13

Find the directional derivative of the function at the given point in the direction of the vector $\vec{v}$.

$$g(s,t) = s\sqrt{t} \quad (2,4) \quad \vec{v} = 2\hat{i} - \hat{j}$$

$$\begin{aligned}
\overrightarrow{\nabla g} &= \langle \frac{\partial g}{\partial s}, \frac{\partial g}{\partial t} \rangle \\
&= \langle \sqrt{t}, \frac{s}{2\sqrt{t}} \rangle \\
\overrightarrow{\nabla g}(2,4) &= \langle \sqrt{4}, \frac{2}{2\sqrt{4}} \rangle \\
&= \langle 2, \frac{1}{2} \rangle \\
D_{\vec{u}} g &= \overrightarrow{\nabla g} \cdot \vec{u} \\
\vec{u} &= \langle \frac{2}{\sqrt{5}}, \frac{-1}{\sqrt{5}} \rangle \\
&= \langle 2, \frac{1}{2} \rangle \cdot \langle \frac{2}{\sqrt{5}}, \frac{-1}{\sqrt{5}} \rangle \\
&= \frac{4}{\sqrt{5}} - \frac{1}{2\sqrt{5}} \\
&= \frac{7}{2\sqrt{5}}
\end{aligned}$$

Exercise 21

Find the maximum rate of change of f at the given point and the direction in which it occurs.

$$f(x,y) = 4y\sqrt{x} \quad (4,1)$$

$$
\begin{aligned}
\overrightarrow{\nabla f} &= \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle \\
&= \langle \frac{4y}{2\sqrt{x}}, 4\sqrt{x} \rangle \\
&= \langle \frac{2y}{\sqrt{x}}, 4\sqrt{x} \rangle \\
\overrightarrow{\nabla f}(4,1) &= \langle \frac{2(1)}{\sqrt{4}}, 4\sqrt{4} \rangle \\
&= \langle 1, 8 \rangle \\
|\overrightarrow{\nabla f}(4,1)| &= \sqrt{65} \\
\text{unit vector } &= \langle \frac{1}{\sqrt{65}}, \frac{8}{\sqrt{65}} \rangle
\end{aligned}
$$

Exercise 29

Find all points at which the direction of fastest change of the function $f(x, y) = x^2 + y^2 - 2x - 4y$ is $\hat{\imath} + \hat{\jmath}$.

$$
\begin{aligned}
\overrightarrow{\nabla f} &= \langle 2x - 2, 2y - 4 \rangle \\
\overrightarrow{\nabla f} &= c(\hat{\imath} + \hat{\jmath}) \\
2x - 2 &= c = 2y - 4 \\
x - 1 &= y - 2 \\
y &= x + 1
\end{aligned}
$$

Any point on the line $y = x + 1$.

Exercise 43

Find equations of the tangent plane and the normal line to the given surface at the specified point.

$$xy^2z^3 = 8 \quad (2, 2, 1)$$

$$
\begin{aligned}
F(x, y, z) &= xy^2z^3 - 8 = 0 \\
F_x &= y^2z^3 \\
F_x(2, 2, 1) &= 4 \\
F_y &= 2yxz^3 \\
F_y(2, 2, 1) &= 8 \\
F_z &= 3z^2xy^2 \\
F_z(2, 2, 1) &= 24 \\
\Pi : 0 &= F_x(x, y, z)(x - x_0) + F_y(x, y, z)(y - y_0) + F_z(x, y, z)(z - z_0) \\
\Pi : 0 &= 4(x - 2) + 8(y - 2) + 24(z - 1) \\
\Pi : 0 &= 4x + 8y + 24z - 48 \\
\Pi : 0 &= x + 2y + 6z - 12 \\
l &= \begin{cases} x &= 4t + 2 \\ y &= 8t + 2 \\ z &= 24t + 1 \end{cases}
\end{aligned}
$$

If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech