

Multivariable and Vector Calculus: Homework 2

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Exercise 5

Find the cross product $\vec{a} \times \vec{b}$ and verify that it is orthogonal to both a and b .

$$\begin{aligned}\vec{a} &= \frac{1}{2}\hat{i} + \frac{1}{3}\hat{j} + \frac{1}{4}\hat{k} \\ \vec{b} &= \hat{i} + 2\hat{j} - 3\hat{k} \\ \vec{a} \times \vec{b} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ 1 & 2 & -3 \end{vmatrix} = \frac{-3}{2}\hat{i} + \frac{7}{4}\hat{j} + \frac{2}{3}\hat{k} \\ \vec{a} \times \vec{b} \cdot \vec{a} &= \frac{-3}{4} + \frac{7}{12} + \frac{2}{12} = 0 \\ \vec{a} \times \vec{b} \cdot \vec{b} &= \frac{-3}{2} + \frac{14}{4} - 2 = 0\end{aligned}$$

Exercise 7

Find the cross product $\vec{a} \times \vec{b}$ and verify that it is orthogonal to both a and b .

$$\begin{aligned}\vec{a} &= \langle t, 1, \frac{1}{t} \rangle \\ \vec{b} &= \langle t^2, t^2, 1 \rangle \\ \vec{a} \times \vec{b} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ t & 1 & \frac{1}{t} \\ t^2 & t^2 & 1 \end{vmatrix} = \langle 1 - t, 0, t^3 - t^2 \rangle \\ \vec{a} \times \vec{b} \cdot \vec{a} &= t(1 - t) + 0 + \frac{t^3 - t^2}{t} \\ &= t - t^2 + t^2 - t = 0 \\ \vec{a} \times \vec{b} \cdot \vec{b} &= t^2(1 - t) + 0 + t^3 - t^2 = 0\end{aligned}$$

Exercise 11

Find the vector, not with determinants, but by using properties of cross products.

$$\begin{aligned}(\hat{j} - \hat{k}) \times (\hat{k} - \hat{i}) &= (\hat{j} - \hat{k}) \times \hat{k} + (\hat{j} - \hat{k}) \times -\hat{i} \\ &= (\hat{j} \times \hat{k}) + (\hat{j} \times -\hat{i}) + (-\hat{k} \times \hat{k}) + (-\hat{k} \times -\hat{i}) \\ &= \hat{i} + \hat{k} + 0 + \hat{j} = \hat{i} + \hat{j} + \hat{k}\end{aligned}$$

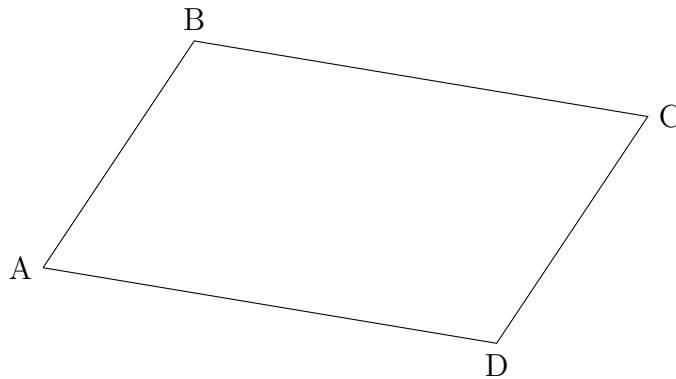
Exercise 13

State whether each expression is meaningful:

1. $\vec{a} \cdot (\vec{b} \times \vec{c})$
Scalar
2. $\vec{a} \times (\vec{b} \cdot \vec{c})$
Nonsensical (vector cross scalar)
3. $\vec{a} \times (\vec{b} \times \vec{c})$
Vector
4. $\vec{a} \cdot (\vec{b} \cdot \vec{c})$
Nonsensical (vector dot scalar)
5. $(\vec{a} \cdot \vec{b}) \times (\vec{c} \cdot \vec{d})$
Nonsensical (scalar cross scalar)
6. $(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d})$
Scalar

Exercise 27

Find the area of the parallelogram with vertices A(-3,0), B(-1,3), C(5,2), and D(3,-1).



$$\begin{aligned}\vec{AB} &= \langle 2, 3 \rangle \\ \vec{AD} &= \langle 6, -1 \rangle \\ \vec{AB} \times \vec{AD} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 0 \\ 6 & -1 & 0 \end{vmatrix} = \langle 0, 0, -20 \rangle\end{aligned}$$

$$\text{area of ABCD} = |\langle 0, 0, -20 \rangle| = 20$$

Exercise 29

Find a nonzero vector orthogonal to the plane through the points P, Q, and R and find the area of $\triangle PQR$:
 $P(1,0,1)$, $Q(-2,1,3)$, $R(4,2,5)$.

$$\begin{aligned}\overrightarrow{PQ} &= \langle -3, 1, 2 \rangle \\ \overrightarrow{PR} &= \langle 3, 2, 4 \rangle \\ \overrightarrow{PQ} \times \overrightarrow{PR} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -3 & 1 & 2 \\ 3 & 2 & 4 \end{vmatrix} = \langle 0, 18, 9 \rangle\end{aligned}$$

$\langle 0, 2, 1 \rangle$ is a vector orthogonal to the plane and the area of $\triangle PQR$ is:

$$\frac{1}{2} |\langle 0, 18, 9 \rangle| = \frac{\sqrt{405}}{2}$$

Exercise 33

Find the volume of the parallelepiped determined by the vectors:

$$\vec{a} = \langle 1, 2, 3 \rangle \quad \vec{b} = \langle -1, 1, 2 \rangle \quad \vec{c} = \langle 2, 1, 4 \rangle$$

$$\begin{aligned}V &= (\vec{a} \times \vec{b}) \cdot \vec{c} \\ &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ -1 & 1 & 2 \end{vmatrix} \cdot \langle 2, 1, 4 \rangle \\ &= \langle 1, -5, 3 \rangle \cdot \langle 2, 1, 4 \rangle \\ &= 2 - 5 + 12 = 9\end{aligned}$$

Exercise 37

Use the scalar triple product to verify that the vectors $\vec{u} = \hat{i} + 5\hat{j} - 2\hat{k}$, $\vec{v} = 3\hat{i} - \hat{j}$, and $w = 5\hat{i} + 9\hat{j} - 4\hat{k}$ are coplanar.

$$\begin{aligned}V &= (\vec{u} \times \vec{v}) \cdot \vec{w} \\ &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 5 & -2 \\ 3 & -1 & 0 \end{vmatrix} \cdot \langle 5, 9, -4 \rangle \\ &= \langle -2, -6, -16 \rangle \cdot \langle 5, 9, -4 \rangle \\ &= -10 - 54 + 64 = 0\end{aligned}$$

Since the scalar triple product is 0, the vectors must be coplanar.

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Find a vector equation and parametric equations for the line through the point $(1,0,6)$ and perpendicular to the plane $x + 3y + z = 5$.

$$\vec{n} = \langle 1, 3, 1 \rangle$$

$$\begin{aligned}\vec{r} &= \langle 1, 0, 6 \rangle + \langle 1, 3, 1 \rangle t \\ &= \langle 1 + t, 3t, 6 + t \rangle \\ &\equiv \begin{cases} x &= 1 + t \\ y &= 3t \\ z &= 6 + t \end{cases}\end{aligned}$$

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Find parametric equations and symmetric equations for the line through the points $(-8, 1, 4)$ and $(3, -2, 4)$.

$$\vec{d} = \langle 11, -3, 0 \rangle$$

$$\begin{aligned}l &= \begin{cases} x &= 3 + 11t \\ y &= -2 - 3t \\ z &= 4 \end{cases} \\ &\equiv \frac{x - 3}{11} = \frac{y + 2}{-3}\end{aligned}$$

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Determine whether the lines L_1 and L_2 are parallel, skew, or intersecting. If they intersect, find the point of intersection.

$$\begin{aligned}L_1 &= \begin{cases} x &= 3 + 2t \\ y &= 4 - t \\ z &= 1 + 3t \end{cases} \\ L_2 &= \begin{cases} x &= 1 + 4s \\ y &= 3 - 2s \\ z &= 4 + 5s \end{cases} \vec{n}_1 = \langle 2, -1, 3 \rangle \\ \vec{n}_2 &= \langle 4, -2, 5 \rangle \\ 3 + 2t &= 1 + 4s \\ t &= 2s - 1 \\ 4 - t &= 3 - 2s \\ 4 - (2s - 1) &= 3 - 2s \\ 5 - 2s &= 3 - 2s\end{aligned}$$

No solution, therefore the lines are skew.

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Find an equation of the plane through the point $(1, -1, -1)$ and parallel to the plane $5x - y - z = 6$.

$$\begin{aligned}\vec{n} &= \langle 5, -1, -1 \rangle \\ \Pi &\equiv 5(x - 1) - 1(y + 1) - 1(z + 1) = 0 \\ &\equiv 5x - 5 - y - 1 - z - 1 = 0 \\ &\equiv 5x - y - z = 7\end{aligned}$$

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Find an equation of the plane through the points $(0,1,1)$, $(1,0,1)$, and $(1,1,0)$.

$$\begin{aligned}\vec{u} &= \langle 1, -1, 0 \rangle \\ \vec{v} &= \langle 1, 0, -1 \rangle \\ \vec{n} &= \vec{u} \times \vec{v} \\ &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{vmatrix} = \langle 1, 1, 1 \rangle \\ \Pi &\equiv 1(x-0) + 1(y-1) + 1(z-1) = 0 \\ &\equiv x + y - 1 + z - 1 = 0 \\ &\equiv x + y + z = 2\end{aligned}$$

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Find the equation of the plane that passes through the point $(3,1,4)$ and contains the line of intersection of the planes $x + 2y + 3z = 1$ and $2x - y + z = -3$.

Let $z = 0$:

$$\begin{aligned}x + 2y &= 1 \\ 2x - y &= -3 \\ x &= 1 - 2y \\ 2(1 - 2y) - y &= -3 \\ 2 - 4y - y &= -3 \\ 2 - 5y &= -3 \\ -5y &= -5 \\ y &= 1 \\ x &= -1\end{aligned}$$

Let $y = 0$:

$$\begin{aligned}x + 3z &= 1 \\ 2x + z &= -3 \\ x &= 1 - 3z \\ 2(1 - 3z) + z &= -3 \\ 2 - 6z + z &= -3 \\ -5z &= -5 \\ z &= 1 \\ x &= -2\end{aligned}$$

Points $(-1,1,0)$ and $(-2,0,1)$:

$$\vec{u} = (3, 1, 4) - (-1, 1, 0) = \langle 4, 0, 4 \rangle$$

$$\vec{v} = (3, 1, 3) - (-2, 0, 1) = \langle 5, 1, 2 \rangle$$

$$\vec{n} = \vec{u} \times \vec{v}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 0 & 4 \\ 5 & 1 & 2 \end{vmatrix} = \langle -4, 12, 4 \rangle$$

$$\Pi \equiv -4(x - (-1)) + 12(y - 1) + 4(z - 0) = 0$$

$$\equiv -4x - 4 + 12y - 12 + 4z = 0$$

$$\equiv -4x + 12y + 4z = 16$$

$$\equiv -x + 3y + z = 4$$

$$\equiv x - 3y - z = -4$$

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Use the formula to find the distance from the point to the given line. $P = (4,1,-2)$

$$l = \begin{cases} x &= 1 + t \\ y &= 3 - 2t \\ z &= 4 - 3t \end{cases}$$

$$\vec{u} = \langle 1, -2, -3 \rangle$$

$$P_0 = (1, 3, 4)$$

$$\vec{v} = \overrightarrow{P_0P} = \langle 3, -2, -6 \rangle$$

$$d = \frac{|\vec{u} \times \vec{v}|}{|\vec{u}|}$$

$$= \frac{\left| \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & -3 \\ 3 & -2 & -6 \end{vmatrix} \right|}{|\langle 1, -2, -3 \rangle|}$$

$$= \frac{|\langle 6, -3, 4 \rangle|}{\sqrt{1 + 4 + 9}}$$

$$= \frac{\sqrt{36 + 9 + 16}}{\sqrt{14}}$$

$$= \sqrt{\frac{61}{14}}$$

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Find the distance from the point to the given plane. $P_1 = (1, -2, 4)$

$$3x + 2y + 6z = 5$$

$$\begin{aligned}
D &= \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}} \\
&= \frac{|3(1) + 2(-2) + 6(4) + (-5)|}{\sqrt{9 + 4 + 36}} \\
&= \frac{|3 - 4 + 24 - 5|}{\sqrt{49}} \\
&= \frac{18}{7}
\end{aligned}$$

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Find the distance between the given parallel planes.

$$2x - 3y + z = 4 \quad 4x - 6y + 2z = 3$$

$$\begin{aligned}
P_0 &= \langle 0, 0, 4 \rangle \\
D &= \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}} \\
&= \frac{|4(0) - 6(0) + 2(4) - 3|}{\sqrt{16 + 36 + 4}} \\
&= \frac{|8 - 3|}{\sqrt{56}} \\
&= \frac{5}{4\sqrt{14}}
\end{aligned}$$

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Show that the lines with symmetric equations $x = y = z$ and $x + 1 = \frac{y}{2} = \frac{z}{3}$ are skew, and find the distance between these lines.

$$l_1 = \begin{cases} x = t \\ y = t \\ z = t \end{cases}$$

$$l_2 = \begin{cases} x = -1 + s \\ y = 2s \\ z = 3s \end{cases}$$

$$t = s - 1$$

$$s - 1 = 2s$$

$$s = -1$$

$$t \stackrel{?}{=} 3s$$

$$s - 1 = 3s$$

$$s = \frac{-1}{2}$$

Since the solutions do not match, the lines are skew.

$$\vec{v}_1 = \langle 1, 1, 1 \rangle$$

$$\vec{v}_2 = \langle 1, 2, 3 \rangle$$

$$\vec{n} = \vec{v}_1 \times \vec{v}_2$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{vmatrix} = \langle 1, -2, 1 \rangle$$

$$P_0 = (0, 0, 0)$$

$$\Pi = 1x - 2y + 1z = 0$$

$$P_1 = (-1, 0, 0)$$

$$\begin{aligned} D &= \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}} \\ &= \frac{|1(-1) + 0 + 0 + 0|}{\sqrt{1 + 4 + 1}} \\ &= \frac{1}{\sqrt{6}} \end{aligned}$$

If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech