Multivariable and Vector Calculus

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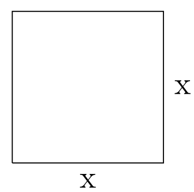
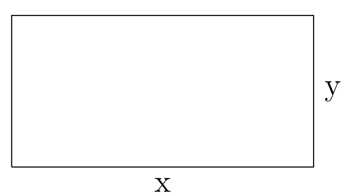
Functions of Several Variables

A function of a single variable:



$$A(x) = x^2$$

A function of two variables:



$$A(x, y) = xy$$

Even for functions of many variables, they still have properties such as domain and range.

$$f(x, y) = \frac{1}{\sqrt{1 - x^2 - y^2}}$$

Domain:

$$1 - x^2 - y^2 > 0 \therefore x^2 + y^2 < 1$$

Range:

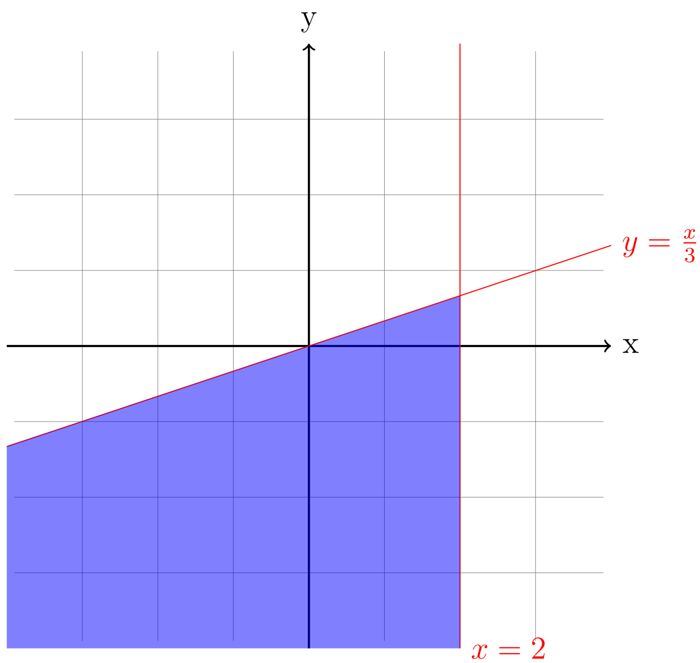
$$[1, \infty)$$

Example

Find the domain of the function:

$$f(x, y) = \frac{\ln(2 - x)}{\sqrt{x - 3y}}$$

$$\begin{aligned} 2 - x &> 0 \\ &\equiv x < 2 \\ x - 3y &> 0 \\ &\equiv y < \frac{x}{3} \end{aligned}$$



Limits

Example

$$\lim_{(x,y)\to(2,1)} \frac{xy}{x^2 + 2y^2} = \frac{1}{3}$$

Example

$$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{1-x^2+y^2}-1} = \lim_{(x,y)\to(0,0)} \frac{(x^2+y^2)(\sqrt{1-x^2-y^2+1})}{(\sqrt{1-x^2-y^2}-1)(\sqrt{1-x^2-y^2}+1)}$$
$$= \lim_{(x,y)\to(0,0)} \frac{(x^2+y^2)(\sqrt{1-x^2-y^2+1})}{1-x^2-y^2+1}$$
$$= -2$$

Definition

$$lim_{(x,y)\to(x_\circ,y_\circ)} = L$$

is defined as for every $\epsilon > 0$ there exists $\delta > 0$ such that $|f(x,y) - L| < \epsilon$ if $dist((x,y),(x_\circ,y_\circ)) < \delta$. Notice that if:

$$\lim_{(x,y)\to(x_\circ,y_\circ)} f(x,y) = L \quad \text{and} \quad \lim_{(x,y)\to(x_\circ,y_\circ)} g(x,) = M$$

Then:

$$\lim_{(x,y)\to(x_\circ,y_\circ)} \frac{f(x,y)}{g(x,y)} = \frac{L}{M}$$

Theorem

If:

$$\lim_{(x,y)\to(x_\circ,y_\circ)} f(x,y) \text{ on } C_1 \neq \lim_{(x,y)\to(x_\circ,y_\circ)} f(x,y) \text{ on } C_1$$

Then $\lim_{(x,y)\to(x_\circ,y_\circ)} f(x,y)$ on C_1 does not exist.

Example

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+2y^2} = \lim_{(x,y)\to(0,0)} \frac{0}{x^2} = \lim_{(x,y)\to(0,0)} \frac{0}{2y^2} = 0$$

Inconclusive according to the theorem above. Take $y = kx$:

$$\lim_{(x,y)\to(x_\circ,y_\circ)} \frac{xkx}{x^2+2kx^2} = \lim_{(x,y)\to(x_\circ,y_\circ)} \frac{k}{1+2k} = \frac{k}{1+2k}$$

The original limit does not exist.

Example

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+2y^2}$$

Take $y = kx$:

$$\lim_{x\to 0} \frac{x^2kx}{x^2+2k^2x^2} = \lim_{x\to 0} \frac{kx}{1+2k^2} = 0$$

Take $y = kx^2$:

$$\lim_{x\to 0} \frac{x^2kx^2}{x^2+2k^2x^4} = \lim_{x\to 0} \frac{kx^2}{1+2k^2x^2} = 0$$

By the Squeeze Theorem:

$$\lim_{(x,y)\to(0,0)} 0 \le \lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+2y^2} \le \lim_{(x,y)\to(0,0)} y$$

We can conclude that $\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+2y^2} = 0$.

Partial Derivatives

$$\lim_{h\to 0} \frac{f(x+h,h_\circ) - f(x,y_\circ)}{h} = \frac{\partial f}{\partial x}(x,y_\circ)$$

Examples:

$$\frac{\partial}{\partial y}(x^2y^3 - x^2 + xy) = x^2 3y^2 + x$$

$$\frac{\partial}{\partial x}(x^2y^3 - x^2 + xy) = y^3 2x - 2x + y$$

$$\frac{\partial}{\partial x}(x^y) = yx^{y-1}$$

$$\frac{\partial}{\partial y}(x^y) = x^y \ln(x)$$

Extensions:

$$\frac{\partial}{\partial x}\frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial x^2}$$

$$\frac{\partial}{\partial y}\frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial y^2}$$

$$\frac{\partial}{\partial y}\frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial y \partial x}$$

$$\frac{\partial}{\partial x}\frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial x \partial y}$$

Clairaut's Theorem

If $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ are continuous in some neighborhood of $(x_\circ, y_\circ)$, then:

$$\frac{\partial^2}{\partial x \partial y}(x_\circ, y_\circ) = \frac{\partial^2}{\partial y \partial x}(x_\circ, y_\circ)$$

$z = f(x, y)$ is called differentiable at $x_\circ, y_\circ$ if and only if $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ exist and are continouous at $(x_\circ, y_\circ)$.

Example

Check that $u(x, t) = \sin(x - at)$ is a solution of:

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

$$\begin{aligned}
\frac{\partial^2 u}{\partial t^2}(\sin(x - at)) &= \frac{\partial u}{\partial t} \cos(x - at)(-a) \\
&= -\sin(x - at)(-a)(-a) \\
&= -a^2 \sin(x - at) \\
a^2 \frac{\partial^2 u}{\partial x^2}(\sin(x - at)) &= a^2 \frac{\partial u}{\partial x} \cos(x - at) \\
&= a^2(-\sin(x - at)) \\
&= -a^2 \sin(x - at)
\end{aligned}$$

Both derivatives are equal.

Example

The temperature of a region is determined by:

$$T(t) = \frac{60}{1 + x^2 + y^2}$$

A bug is located at (2,1), in which direction should it go to cool off.

$$\begin{aligned}
\frac{\partial T}{\partial x} &= \frac{-20}{3} \\
\frac{\partial T}{\partial y} &= \frac{-10}{3}
\end{aligned}$$

The bug should move towards the north and east.

Example

$$z = f(x, y) \quad x = x(t) \quad y = y(t)$$

The function z can be defined in terms of t: $z(t)$

$$\begin{aligned}
\frac{\mathrm{d}z}{\mathrm{d}t} &= \lim_{h\to 0} \frac{z(t+h) - z(t)}{h} \\
&= \lim_{h\to 0} \frac{f(x(t+h), y(t+h)) - f(x(t), y(t))}{h} \\
&= \lim_{h\to 0} \frac{(f(x(t+h), y(t+h)) - f(x(t+h), y(t)))(y(t+h) - y(t))}{h(y(t+h) - y(t))} \\
&+ \frac{(f(x(t+h), y(t)) - f(x(t), y(t)))(x(t+h) - x(t))}{h(x(t+h) - x(t))} \\
&\lim_{h\to 0} \frac{(x(t+h) - x(t))}{h} = \frac{\mathrm{d}x}{\mathrm{d}t} \\
&\lim_{h\to 0} \frac{(y(t+h) - y(t))}{h} = \frac{\mathrm{d}y}{\mathrm{d}t} \\
&\lim_{h\to 0} \frac{(f(x(t+h), y(t)) - f(x(t), y(t)))}{(x(t+h) - x(t))} = \frac{\partial f}{\partial x} \\
\frac{\mathrm{d}z}{\mathrm{d}t} &= \frac{\partial z}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}
\end{aligned}$$

Example

$$W = W(T, R) \quad \frac{\partial W}{\partial T} = -2 \quad \frac{\partial W}{\partial R} = 4$$

$$\frac{\mathrm{d}T}{\mathrm{d}t} = 0.1 \quad \frac{\mathrm{d}R}{\mathrm{d}t} = 0.5$$

$$\frac{\mathrm{d}W}{\mathrm{d}t} = \frac{\partial W}{\partial T}\frac{\mathrm{d}T}{\mathrm{d}t} + \frac{\partial W}{\partial R}\frac{\mathrm{d}R}{\mathrm{d}t} = 1.8$$

Example

Suppose you have a simple electric circuit:

$$V = IR \quad R = 4.00\Omega \quad I = 0.08A \quad \frac{\mathrm{d}R}{\mathrm{d}t} = 0.03\frac{\Omega}{s} \quad \frac{\mathrm{d}V}{\mathrm{d}t} = -0.01V$$

What is $\frac{\mathrm{d}I}{\mathrm{d}t}$?

$$\begin{aligned}
I(V, R) &= \frac{\mathrm{d}I}{\mathrm{d}t} \\
&= \frac{\partial I}{\partial V}\frac{\mathrm{d}V}{\mathrm{d}t} + \frac{\partial I}{\partial R}\frac{\mathrm{d}R}{\mathrm{d}t} \\
&= \frac{1}{R}(-0.01) + \frac{-V}{R^2}0.03 \\
&= \frac{1}{400}(-0.01) - \frac{0.08}{400}(0.03)
\end{aligned}$$

Example

Suppose you have the surface

$$x^3y^2z^4 + 4xy + z^2 - 6 = 0$$

What is $\frac{\mathrm{d}z}{\mathrm{d}y}$?

$$\begin{aligned}
F(x, y, z(x, y)) &= 0 \\
\frac{\mathrm{d}z}{\mathrm{d}y}F(x, y, z(x, y)) &= \frac{\partial F}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}y} + \frac{\partial F}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}y} + \frac{\partial F}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}y} = 0 \\
\frac{\mathrm{d}x}{\mathrm{d}y} &= 0 \\
\frac{\mathrm{d}y}{\mathrm{d}y} &= 1 \\
\therefore \frac{\mathrm{d}z}{\mathrm{d}y} &= \frac{-\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}} \\
&= -\frac{F_y}{F_z} \\
&= -\frac{x^3z^4(2y) + 4x}{x^3y^2(4z^3) + 2z}
\end{aligned}$$

Example

$$y^2x^4 + y^3x - 2 = 0$$

Find $\frac{\mathrm{d}y}{\mathrm{d}x}$:

$$\begin{aligned} 2y\ \mathrm{d}yxx^4 + y^2 4x^3 + 3y^2 \frac{\mathrm{d}y}{\mathrm{d}x} x + y^3 = 0 \\ \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-y^2 4x^3 - y^3}{2y + 3y^2 x} \end{aligned}$$

Example

$$S : F(x, y, z) = 0 \quad (x_\circ, y_\circ, z_\circ) \in S$$
$$C : r(\vec{t}) = \langle x(t), y(t), z(t) \rangle \quad C \subset S \quad (x_\circ, y_\circ, z_\circ) \in S$$

$$\begin{aligned} \frac{\partial F}{\partial x}\ \mathrm{d}xt + \frac{\partial F}{\partial t}\frac{\partial y}{\partial t} + \frac{\partial F}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t} &= 0 \\ &= \langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \rangle \cdot \langle \frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}y}{\mathrm{d}t}, \frac{\mathrm{d}z}{\mathrm{d}t} \rangle \\ &= \overrightarrow{\nabla F} \cdot \overrightarrow{r'(t)} = 0 \\ &\therefore \overrightarrow{\nabla F} \perp \overrightarrow{r'(t)} \end{aligned}$$

Example

Check that every line normal to a sphere passes through its center.

$$x^2 + y^2 + z^2 - 1 = 0$$

$$\begin{aligned} \vec{u} &= \overrightarrow{\nabla F} \\ &= \langle 2x, 2y, 2z \rangle \\ l_{normal} &= \begin{cases} x &= x_\circ + t2x_\circ \\ y &= y_\circ + t2y_\circ \\ z &= z_\circ + t2z_\circ \end{cases} \\ t &= -\frac{1}{2} \end{aligned}$$

Directional Derivative

As an extension of the gradient definition:

$$D_{\vec{u}}f = \overrightarrow{\nabla f} \cdot \vec{u} = |\overrightarrow{\nabla f}||\vec{u}|\cos\theta \quad |\vec{u}| = 1$$

The maximum value of $D_{\vec{u}}$ is $|\overrightarrow{\nabla f}|$, which occurs when:

$$\vec{u} = \frac{\overrightarrow{\nabla F}}{|\overrightarrow{\nabla f}|}$$

The minimum value of $D_{\vec{u}}$ is $-|\overrightarrow{\nabla f}|$, which occurs when:

$$\vec{u} = -\frac{\overrightarrow{\nabla F}}{|\overrightarrow{\nabla f}|}$$

Example

$$z = x^2 + y^2$$

Find the directional derivative of z at (1,2) in the direction towards (5,5).

$$\begin{aligned} \vec{u} &= \langle 5-1, 5-2 \rangle \\ &= \langle \frac{4}{5}, \frac{3}{5} \rangle \\ D_z(1,2) &= \overrightarrow{\nabla z} \cdot \vec{u} \\ &= \langle 2x, 2y \rangle \cdot \vec{u} \\ &= \langle 2, 4 \rangle \cdot \langle \frac{4}{5}, \frac{3}{5} \rangle \\ &= 4 \end{aligned}$$

You can find all my notes at `http://omgimanerd.tech/notes`. If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech