

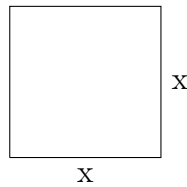
# Multivariable and Vector Calculus

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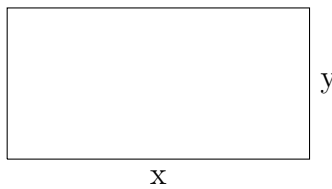
## Functions of Several Variables

A function of a single variable:



$$A(x) = x^2$$

A function of two variables:



$$A(x, y) = xy$$

Even for functions of many variables, they still have properties such as domain and range.

$$f(x, y) = \frac{1}{\sqrt{1 - x^2 - y^2}}$$

Domain:

$$1 - x^2 - y^2 > 0 \therefore x^2 + y^2 < 1$$

Range:

$$[1, \infty)$$

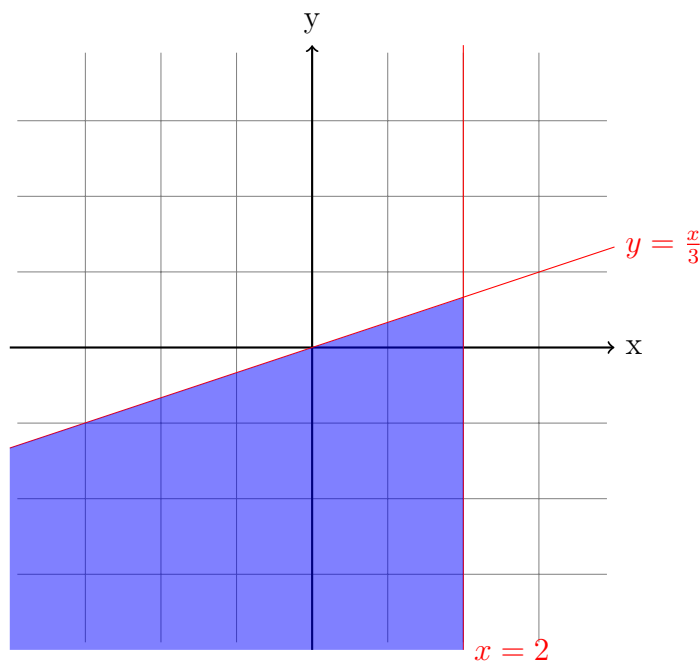
## Example

Find the domain of the function:

$$f(x, y) = \frac{\ln(2-x)}{\sqrt{x-3y}}$$

$$\begin{aligned} 2-x &> 0 \\ &\equiv x < 2 \end{aligned}$$

$$\begin{aligned} x-3y &> 0 \\ &\equiv y < \frac{x}{3} \end{aligned}$$



## Limits

### Example

$$\lim_{(x,y) \rightarrow (2,1)} \frac{xy}{x^2 + 2y^2} = \frac{1}{3}$$

### Example

$$\begin{aligned}\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 + y^2}{\sqrt{1 - x^2 + y^2} - 1} &= \lim_{(x,y) \rightarrow (0,0)} \frac{(x^2 + y^2)(\sqrt{1 - x^2 - y^2 + 1})}{(\sqrt{1 - x^2 - y^2} - 1)(\sqrt{1 - x^2 - y^2 + 1})} \\ &= \lim_{(x,y) \rightarrow (0,0)} \frac{(x^2 + y^2)(\sqrt{1 - x^2 - y^2 + 1})}{1 - x^2 - y^2 + 1} \\ &= -2\end{aligned}$$

### Definition

$$\lim_{(x,y) \rightarrow (x_0, y_0)} f(x, y) = L$$

is defined as for every  $\epsilon > 0$  there exists  $\delta > 0$  such that  $|f(x, y) - L| < \epsilon$  if  $\text{dist}((x, y), (x_0, y_0)) < \delta$ . Notice that if:

$$\lim_{(x,y) \rightarrow (x_0, y_0)} f(x, y) = L \quad \text{and} \quad \lim_{(x,y) \rightarrow (x_0, y_0)} g(x, y) = M$$

Then:

$$\lim_{(x,y) \rightarrow (x_0, y_0)} \frac{f(x, y)}{g(x, y)} = \frac{L}{M}$$

### Theorem

If:

$$\lim_{(x,y) \rightarrow (x_0, y_0)} f(x, y) \text{ on } C_1 \neq \lim_{(x,y) \rightarrow (x_0, y_0)} f(x, y) \text{ on } C_2$$

Then  $\lim_{(x,y) \rightarrow (x_0, y_0)} f(x, y)$  on  $C_1$  does not exist.

### Example

$$\lim_{(x,y) \rightarrow (0,0)} \frac{xy}{x^2 + 2y^2} = \lim_{(x,y) \rightarrow (0,0)} \frac{0}{x^2} = \lim_{(x,y) \rightarrow (0,0)} \frac{0}{2y^2} = 0$$

Inconclusive according to the theorem above. Take  $y = kx$ :

$$\lim_{(x,y) \rightarrow (0,0)} \frac{xkx}{x^2 + 2kx^2} = \lim_{(x,y) \rightarrow (0,0)} \frac{k}{1 + 2k} = \frac{k}{1 + 2k}$$

The original limit does not exist.

### Example

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y}{x^2 + 2y^2}$$

Take  $y = kx$ :

$$\lim_{x \rightarrow 0} \frac{x^2 kx}{x^2 + 2k^2 x^2} = \lim_{x \rightarrow 0} \frac{kx}{1 + 2k^2} = 0$$

Take  $y = kx^2$ :

$$\lim_{x \rightarrow 0} \frac{x^2 kx^2}{x^2 + 2k^2 x^4} = \lim_{x \rightarrow 0} \frac{kx^2}{1 + 2k^2 x^2} = 0$$

By the Squeeze Theorem:

$$\lim_{(x,y) \rightarrow (0,0)} 0 \leq \lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y}{x^2 + 2y^2} \leq \lim_{(x,y) \rightarrow (0,0)} y$$

We can conclude that  $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y}{x^2 + 2y^2} = 0$ .

## Partial Derivatives

$$\lim_{h \rightarrow 0} \frac{f(x+h, y_0) - f(x, y_0)}{h} = \frac{\partial f}{\partial x}(x, y_0)$$

Examples:

$$\frac{\partial}{\partial y}(x^2 y^3 - x^2 + xy) = x^2 3y^2 + x$$

$$\frac{\partial}{\partial x}(x^2 y^3 - x^2 + xy) = y^3 2x - 2x + y$$

$$\frac{\partial}{\partial x}(x^y) = yx^{y-1}$$

$$\frac{\partial}{\partial y}(x^y) = x^y \ln(x)$$

Extensions:

$$\frac{\partial}{\partial x} \frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial x^2}$$

$$\frac{\partial}{\partial y} \frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial y^2}$$

$$\frac{\partial}{\partial y} \frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial y \partial x}$$

$$\frac{\partial}{\partial x} \frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial x \partial y}$$

## Clairaut's Theorem

If  $\frac{\partial^2 f}{\partial x \partial y}$  and  $\frac{\partial^2 f}{\partial y \partial x}$  are continuous in some neighborhood of  $(x_0, y_0)$ , then:

$$\frac{\partial^2}{\partial x \partial y}(x_0, y_0) = \frac{\partial^2}{\partial y \partial x}(x_0, y_0)$$

$z = f(x, y)$  is called differentiable at  $x_0, y_0$  if and only if  $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$  exist and are continuous at  $(x_0, y_0)$ .

### Example

Check that  $u(x, t) = \sin(x - at)$  is a solution of:

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2}(\sin(x - at)) &= \frac{\partial u}{\partial t} \cos(x - at)(-a) \\ &= -\sin(x - at)(-a)(-a) \\ &= -a^2 \sin(x - at) \\ a^2 \frac{\partial^2 u}{\partial x^2}(\sin(x - at)) &= a^2 \frac{\partial u}{\partial x} \cos(x - at) \\ &= a^2(-\sin(x - at)) \\ &= -a^2 \sin(x - at) \end{aligned}$$

Both derivatives are equal.

### Example

The temperature of a region is determined by:

$$T(t) = \frac{60}{1 + x^2 + y^2}$$

A bug is located at  $(2,1)$ , in which direction should it go to cool off.

$$\begin{aligned} \frac{\partial T}{\partial x} &= \frac{-20}{3} \\ \frac{\partial T}{\partial y} &= \frac{-10}{3} \end{aligned}$$

The bug should move towards the north and east.

### Example

$$z = f(x, y) \quad x = x(t) \quad y = y(t)$$

The function  $z$  can be defined in terms of  $t$ :  $z(t)$

$$\begin{aligned} \frac{dz}{dt} &= \lim_{h \rightarrow 0} \frac{z(t+h) - z(t)}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(x(t+h), y(t+h)) - f(x(t), y(t))}{h} \\ &= \lim_{h \rightarrow 0} \frac{(f(x(t+h), y(t+h)) - f(x(t+h), y(t)))(y(t+h) - y(t))}{h(y(t+h) - y(t))} \\ &\quad + \frac{(f(x(t+h), y(t)) - f(x(t), y(t)))(x(t+h) - x(t))}{h(x(t+h) - x(t))} \\ &\lim_{h \rightarrow 0} \frac{(x(t+h) - x(t))}{h} = \frac{dx}{dt} \\ &\lim_{h \rightarrow 0} \frac{(y(t+h) - y(t))}{h} = \frac{dy}{dt} \\ &\lim_{h \rightarrow 0} \frac{(f(x(t+h), y(t)) - f(x(t), y(t)))}{(x(t+h) - x(t))} = \frac{\partial f}{\partial x} \\ \frac{dz}{dt} &= \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} \end{aligned}$$

### Example

$$\begin{aligned} W = W(T, R) \quad \frac{\partial W}{\partial T} = -2 \quad \frac{\partial W}{\partial R} = 4 \\ \frac{dT}{dt} = 0.1 \quad \frac{dR}{dt} = 0.5 \\ \frac{dW}{dt} = \frac{\partial W}{\partial T} \frac{dT}{dt} + \frac{\partial W}{\partial R} \frac{dR}{dt} = 1.8 \end{aligned}$$

### Example

Suppose you have a simple electric circuit:

$$V = IR \quad R = 4.00\Omega \quad I = 0.08A \quad \frac{dR}{dt} = 0.03 \frac{\Omega}{s} \quad \frac{dV}{dt} = -0.01V$$

What is  $\frac{dI}{dt}$ ?

$$\begin{aligned} I(V, R) &= \frac{dI}{dt} \\ &= \frac{\partial I}{\partial V} \frac{dV}{dt} + \frac{\partial I}{\partial R} \frac{dR}{dt} \\ &= \frac{1}{R}(-0.01) + \frac{-V}{R^2} 0.03 \\ &= \frac{1}{400}(-0.01) - \frac{0.08}{400}(0.03) \end{aligned}$$

### Example

Suppose you have the surface

$$x^3 y^2 z^4 + 4xy + z^2 - 6 = 0$$

What is  $\frac{dz}{dy}$ ?

$$\begin{aligned} F(x, y, z(x, y)) &= 0 \\ \frac{dz}{dy} F(x, y, z(x, y)) &= \frac{\partial F}{\partial x} \frac{dx}{dy} + \frac{\partial F}{\partial y} \frac{dy}{dy} + \frac{\partial F}{\partial z} \frac{dz}{dy} = 0 \\ \frac{dx}{dy} &= 0 \\ \frac{dy}{dy} &= 1 \\ \therefore \frac{dz}{dy} &= \frac{-\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}} \\ &= -\frac{F_y}{F_z} \\ &= -\frac{x^3 z^4 (2y) + 4x}{x^3 y^2 (4z^3) + 2z} \end{aligned}$$

### Example

$$y^2 x^4 + y^3 x - 2 = 0$$

Find  $\frac{dy}{dx}$ :

$$2ydyxx^4 + y^24x^3 + 3y^2\frac{dy}{dx}x + y^3 = 0$$
$$\frac{dy}{dx} = \frac{-y^24x^3 - y^3}{2y + 3y^2x}$$

**Example**

$$S : F(x, y, z) = 0 \quad (x_o, y_o, z_o) \in S$$

$$C : \vec{r}(t) = \langle x(t), y(t), z(t) \rangle \quad C \subset S \quad (x_o, y_o, z_o) \in S$$

$$\begin{aligned} \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial t} \frac{dy}{dt} + \frac{\partial F}{\partial z} \frac{dz}{dt} &= 0 \\ &= \left\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle \\ &= \vec{\nabla} F \cdot \vec{r}'(t) = 0 \\ \therefore \vec{\nabla} F &\perp \vec{r}'(t) \end{aligned}$$

**Example**

Check that every line normal to a sphere passes through its center.

$$x^2 + y^2 + z^2 - 1 = 0$$

$$\begin{aligned} \vec{u} &= \vec{\nabla} F \\ &= \langle 2x, 2y, 2z \rangle \\ l_{normal} &= \begin{cases} x &= x_o + t2x_o \\ y &= y_o + t2y_o \\ z &= z_o + t2z_o \end{cases} \\ t &= -\frac{1}{2} \end{aligned}$$



## Directional Derivative

As an extension of the gradient definition:

$$D_{\vec{u}}f = \vec{\nabla}f \cdot \vec{u} = |\vec{\nabla}f||\vec{u}| \cos \theta \quad |\vec{u}| = 1$$

The maximum value of  $D_{\vec{u}}$  is  $|\vec{\nabla}f|$ , which occurs when:

$$\vec{u} = \frac{\vec{\nabla}f}{|\vec{\nabla}f|}$$

The minimum value of  $D_{\vec{u}}$  is  $-|\vec{\nabla}f|$ , which occurs when:

$$\vec{u} = -\frac{\vec{\nabla}f}{|\vec{\nabla}f|}$$

### Example

$$z = x^2 + y^2$$

Find the directional derivative of  $z$  at  $(1,2)$  in the direction towards  $(5,5)$ .

$$\begin{aligned}\vec{u} &= \langle 5 - 1, 5 - 2 \rangle \\ &= \left\langle \frac{4}{5}, \frac{3}{5} \right\rangle \\ D_z(1, 2) &= \vec{\nabla}z \cdot \vec{u} \\ &= \langle 2x, 2y \rangle \cdot \vec{u} \\ &= \langle 2, 4 \rangle \cdot \left\langle \frac{4}{5}, \frac{3}{5} \right\rangle \\ &= 4\end{aligned}$$

You can find all my notes at <http://omgimanerd.tech/notes>. If you have any questions, comments, or concerns, please contact me at [alvin@omgimanerd.tech](mailto:alvin@omgimanerd.tech)