Multivariable and Vector Calculus

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Vector Functions and Space Curves

$$v : t \in \mathbb{R} \longrightarrow v(\vec{t}) = \langle x(t), y(t), z(t) \rangle$$

Parametric line equations are similar to this form:

$$l = \begin{cases} x &= x_\circ + ta_1 \\ y &= y_\circ + ta_2 \\ z &= z_\circ + ta_3 \end{cases}$$

$$r(t) = \langle x_\circ + ta_1, y_\circ + ta_2, z_\circ + ta_3 \rangle$$

Example

$$v(\vec{t}) = \langle \cos(t), \sin(t), t^3 \rangle$$

We know that $x = \cos(t)$ and $y = \sin(t)$ and we have the identity $\sin^2\theta + \cos^2\theta = 1$, therefore $x^2 + y^2 = 1$. We can determine this this vector traces a helix on the perimeter of a cylinder along the z-axis.

Example

Find the point where the helix $v(t) = \langle \cos(t), \sin(t), t \rangle$ intersects the sphere with equation $x^2 + y^2 + z^2 = 5$.

$$\begin{aligned} x^2 + y^2 + z^2 &= 5 \\ \cos^2 t + \sin^2 t + t^2 &= 5 \\ 1 + t^2 &= 5 \\ t^2 &= 4 \\ t &= \pm 2 \end{aligned}$$

There are two points at which they intersect, when $t = 2$ and when $t = -2$, $P_1(\cos 2, \sin(2), 2)$ and $P_2(\cos(-2), \sin(-2), -2)$.

Example

Suppose you have two surfaces $S_1 : x^2 + y^2 = 4$ and $S_2 : z = xy$. Find the curve of intersection C. We can let $x = 2\cos(t)$ and $y = 2\sin(t)$ in order to fulfill $x^2 + y^2 = 4$.

$$r(\vec{t}) = \langle 2\cos(t), 2\sin(t), 4\cos(t)\sin(t)$$

Example

Given the vector equation for the curve of intersection between $S_1 : z = 4x^2 + y^2$ and $S_2 : y = x^2$. In terms of x:

$$r(\vec{x}) = \langle x, x^2, 4x^2 + y^2\rangle = \langle x, x^2, 4x^2 + x^4\rangle$$

$$r(\vec{t}) = \langle t, t^2, 4t^2 + t^4\rangle$$

Example

Give the vector equation for the curve of intersection between $S_1 : x^2 + z^2 = 1$ and $S_2 : x^2 + y^2 + 4z^2 = 4, y \geq 0$.

$$r(\vec{t}) = \langle \cos(t), \sqrt{4 - \cos^2(t) - 4\sin^2(t)}, \sin(t)\rangle$$

Example

Given a surface S such that the curve $r(\vec{t}) = \langle t^2, \ln(t), \frac{1}{2}\rangle$ belongs to S.

$$t^2 - t^2 = 0$$

$$x - e^{2\ln(t)} = 0$$

Example

Find the length of the arc of the curve $\overrightarrow{r(t)} = \langle t^2, 9t, 4t^{\frac{3}{2}}\rangle \quad 1 \leq t \leq 4$:

$$\begin{aligned} L &= \int_1^4 |\langle 2t, 9, 4\frac{3}{2}\sqrt{t}\rangle| \text{ d}t \\ &= \int_1^4 \sqrt{4t^2 + 81 + 36t} \text{ d}t \\ &= \int_1^4 2t + 9 \text{ d}t \\ &= t^2 + 9\Big]_1^4 \end{aligned}$$

Example

Find a point on $\overrightarrow{r(t)} = \langle 3\sin(t), 4t, 3\cos(t)\rangle$ 5 units from (0,0,3) in the direction of increasing t.

$$\begin{aligned} L &= 5 \\ &= \int_0^\tau |\langle 3\cos(t), 4, -3\sin(t)\rangle| \text{ d}t \\ &= \int_0^\tau \sqrt{9\cos^2(t) + 16 + 9\sin^2(t)} \text{ d}t \\ &= \int_0^\tau \sqrt{9(\cos^2(t) + \sin^2(t))16} \text{ d}t \\ &= \int_0^\tau \sqrt{9 + 16} \text{ d}t \\ &= \int_0^\tau 5 \text{ d}t \\ &= 5t\Big]_{t=0}^{t=\tau} \\ &= 5\tau \\ \tau &= 1 \\ P &= \big(3\sin(1), 4, 3\cos(1)\big) \end{aligned}$$

Example

Given:

$$\begin{aligned}
\overrightarrow{v(0)} &= \langle 1, 0, -4 \rangle \\
\overrightarrow{v'(0)} &= \langle 0, 0, -1 \rangle \\
\overrightarrow{v''(t)} &= \langle \sin(t), 2\cos(t), 6t \rangle
\end{aligned}$$

Find $\overrightarrow{v(t)}$:

$$\begin{aligned}
\overrightarrow{v'(t)} &= \langle -\cos(t) + c_1, 2\sin(t) + c_2, 3t^2 + c_3 \rangle = \langle 0, 0, -1 \rangle \\
&\therefore \quad c_1 = 1 \quad c_2 = 0 \quad c_3 = -1 \\
\overrightarrow{v'(t)} &= \langle -\cos(t) + 1, 2\sin(t), 3t^2 - 1 \rangle \\
\overrightarrow{v(t)} &= \langle -\sin(t) + t + c_1, -2\cos(t) + c_2, t^3 - t + c_3 \rangle = \langle 1, 0, -4 \rangle \\
&\therefore \quad c_1 = 1 \quad c_2 = 2 \quad c_3 = -4 \\
\overrightarrow{v(t)} &= \langle -\sin(t) + t + 1, -2\cos(t) + 2, t^3 - t - 4 \rangle
\end{aligned}$$

Example

A bullet fired at an angle of 30° to the horizontal lands 1600ft away. Find $v_\circ$.
Newton's Second Law:

$$\vec{F} = m\vec{a}$$
$$\equiv m\vec{v''} = \langle 0, -mg\rangle$$
$$\text{Assume } m = 1$$
$$\equiv \overrightarrow{v''(t)} = \langle 0, -g\rangle$$
$$= \langle c_1, -gt + c_2\rangle = \langle v_\circ \cos(30), v_\circ \cos(60)\rangle \text{ at } t = 0$$
$$\therefore \quad c_1 = \frac{v_\circ\sqrt{3}}{2} \quad c_2 = \frac{v_\circ}{2}$$
$$\overrightarrow{v''(t)} = \langle \frac{v_\circ\sqrt{3}}{2}, -gt + \frac{v_\circ}{2}\rangle$$
$$\overrightarrow{v(t)} = \langle \frac{v_\circ\sqrt{3}}{2}t + c_1, -gt + \frac{v_\circ t}{2} + c_2\rangle = \langle 0, 0\rangle$$
$$\therefore \quad c_1 = 0 \quad c_2 = 0$$
$$\overrightarrow{v(t)} = \langle \frac{v_\circ\sqrt{3}}{2}t, \frac{-gt^2}{2} + \frac{v_\circ}{2}t\rangle = \langle 0, 0\rangle$$
$$\frac{-gt^2}{2} + \frac{v_\circ}{2}t = 0$$
$$\therefore t = \frac{v_\circ}{g}$$
$$\frac{v_\circ\sqrt{3}}{2}t = 1600$$
$$\frac{v_\circ\sqrt{3}}{2}\frac{v_\circ}{g} = 1600$$
$$v_\circ = \sqrt{\frac{3200g}{\sqrt{3}}}$$

You can find all my notes at `http://omgimanerd.tech/notes`. If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech