Multivariable and Vector Calculus

Alvin Lin

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Describing a line in $\mathbb{R}^3$

Consider a line l such that a point $P_0 \in l, l \parallel \vec{a}$. To describe the line, for any point $P \in l$:

$$\overrightarrow{P_0P} \parallel \vec{a} \equiv \vec{r} - \vec{r_0} = t\vec{a}$$

where $\vec{r}$ is the vector from the origin to the point P and $\vec{r_0}$ is the vector from the origin to the point P_0.

$$\begin{aligned}
\vec{r} &= \vec{r_0} + t\vec{a} \\
\langle x, y, z\rangle &= \langle x_0, y_0, z_0\rangle + t\langle a_1, a_2, z_3\rangle \\
x &= x_0 + ta_1 \\
y &= y_0 + ta_2 \\
z &= z_0 + ta_3 \\
\frac{x - x_0}{a_1} &= \frac{y - y_0}{a_2} = \frac{z - z_0}{a_3}
\end{aligned}$$

Example

Line $l : P_0(1, 1, 2), P_1(2, 3, 5) \in l$

$$\begin{aligned}
\vec{r} &= \langle 1, 1, 2\rangle + t\langle 2 - 1, 3 - 1, 5 - 2\rangle \\
&= \langle 1, 2, 3\rangle \\
x &= 1 + t \\
y &= 1 + 2t \\
z &= 2 + 3t
\end{aligned}$$

Example

There is a bug B traveling on l_1:

$$l_1 = \begin{cases} x &= 2+t \\ y &= 4+2t \\ z &= 1+t \end{cases}$$

Another bug b is traveling on l_2:

$$l_2 = \begin{cases} x &= 1+t \\ y &= 4+t \\ z &= 4-t \end{cases}$$

Do their paths intersect at the same time?

$$\begin{aligned} x &= 2+t = 1+t \\ y &= 4+2t = 4+t \\ z &= 1+t = 4-t \end{aligned}$$

Since there is no solution, they do not intersect at the same time. Is there a point of intersection between their paths at different times?

$$\begin{aligned} x = 2+t & & = 1+\tau \\ t &= \tau - 1 \\ y = 4+2t &= 4+\tau \\ 4+2(\tau-1) &= 4+\tau \\ 2+2\tau &= 4+\tau \\ \tau &= 2 \\ t &= 1 \\ \\ z = 1+t &= 4-\tau \\ 1+1 &\stackrel{?}{=} 4-2 \\ 2 &= 2 \end{aligned}$$

It is possible to solve for a τ and t, so the two paths do cross.

Example

In $\mathbb{R}^2$ we have the line $y = 2x + 3$, how do we convert this to $\mathbb{R}^3$? Two points on this line are (0,3,0) and (1,5,0):

$$l = \begin{cases} x & = 0 + t(1-0) = t \\ y & = 3 + t(5-3) = 3 + 2t \\ z & = 0 + t(0-0) = 0 \end{cases}$$

Example

Does the point (1,2,-3) lie on the line:

$$\frac{x-1}{3} = \frac{y+2}{1} = \frac{z-1}{-1}$$

With the point (1,2,-3):

$$0 \neq 4 = 4$$

As a counterexample, (1,-2,1) is a point that lies on the line.

Describing a plane in $\mathbb{R}^3$

Given a plane Π: $P_0 \in \Pi$, $\vec{n} \perp \Pi$:

$$P \in \Pi \equiv (\overrightarrow{P_0P} \perp \vec{n}) \equiv \overrightarrow{P_0P} \cdot \vec{n} = 0$$

where P is any point on plane Π. Plane Π is described by:

$$n_1(x - x_0) + n_2(y - y_0) + n_3(z - z_0) = 0$$

$$n_1 x + n_2 y + n_3 z = 0$$

Example

Suppose the plane Π contains the points A(1,1,1), B(2,1,3), C(1,4,4).

$$\vec{n} = \vec{AB} \times \vec{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 2 \\ 0 & 3 & 3 \end{vmatrix} = \langle 6, -3, 3 \rangle = \langle 2, 1, -1 \rangle$$

$$2(x-1) + 1(y-1) - 1(z-1) = 0$$

$$2x + y - z = 2$$

Example

Given a plane Π described by $2x+y-z=5$, given a plane Π_1 parallel to Π containing the point P_0(1,2,3).

$$\vec{n} = \langle 2, 1, -1 \rangle$$

$$\Pi_2 \equiv 2(x-1) + 1(y-2) - 1(z-3) = 0 \equiv 2x + y - z = 1$$

Example

Given:

$$\Pi_1 \equiv x + y + z = 4$$

$$\Pi_2 \equiv 2x - y + 2x = 5$$

Find the line of intersection between the planes. Suppose we pick $z = 0$:

$$x + y = 4$$

$$2x - y = 5$$

Solving for this, we get the point P_0(3,1,0). If we pick $x = 0$, we get the point P_1(0,1,3). We can use these two points to get the line:

$$l = \begin{cases} x & = 3 + t(0-3) = 3 - 3t \\ y & = 1 + t(1-1) = 1 \\ z & = 0 + t(3-0) = 3t \end{cases}$$

What is the angle θ between Π_0, Π_1? We can compute the angle between the normal vectors of the planes.

$$\begin{aligned} \cos\angle(n_0, n_1) &= \frac{\vec{n_0} \cdot \vec{n_1}}{|n_0||n_1||} \\ &= \frac{\langle 1, 1, 1 \rangle \cdot \langle 2, -1, 2 \rangle}{\sqrt{3}\sqrt{9}} \\ &= \frac{3}{3\sqrt{3}} \\ &= \frac{\sqrt{3}}{3} \\ \theta &= \cos^{-1} \frac{\sqrt{3}}{3} \end{aligned}$$

Example

Given two planes described by:

$$\Pi_1 : x + 2y + z = 4$$
$$\Pi_2 : x + 2x + z = 4$$

Find the distance between the planes. We can pick two arbitrary points on either plane and project the vector between them onto the normal vector. Let P_1 = (4,0,0) and P_2 = (6,0,0).

$$\begin{aligned} dist(\Pi_1, \Pi_2) &= comp_{\vec{n}} \overrightarrow{P_1P_2} \\ &= \frac{\overrightarrow{P_1P_2} \cdot \vec{n}}{|\vec{n}|} \\ &= \frac{\langle 2, 0, 0 \rangle \cdot \langle 1, 2, 1 \rangle}{\|\langle 1, 2, 1 \rangle|} \\ &= \frac{2}{\sqrt{6}} \end{aligned}$$

Alternatively, we can use a point on Π_1 and find the equation of the line perpendicular to that point. We can find the intersection between that line and the Π_2.

$$P_1 \in l, l \perp \Pi_1 : l = \begin{cases} x &= 4 + t \\ y &= 0 + 2t \\ z &= 0 + t \end{cases}$$

$\Pi_2 \cup l$ is the intersection between the perpendicular line and Π_2.

$$\begin{aligned} x + 2y + z &= 6 \\ (4 + t) + 2(2t) + (t) &= 6 \\ 6t + 4 &= 6 \\ t &= \frac{1}{3} \end{aligned}$$

Using this t, we can get the point $(\frac{13}{3}, \frac{2}{3}, \frac{1}{3})$. We can then use the distance formula between this point and (4,0,0) to get the same result.

Example

Extending the previous problem, find Π_3 such that the distance between the planes is 3. There exists a point $(d, 0, 0)$ on such a plane that satisfies this condition.

$$\begin{aligned}
dist(\Pi_1, \Pi_3) &= comp_{\vec{n}} \overrightarrow{P_1P_3} \\
3 &= \frac{\overrightarrow{P_1P_3} \cdot \vec{n}}{|\vec{n}|} \\
&= \frac{\langle d-4, 0, 0\rangle \cdot \langle 1, 2, 1\rangle}{|\langle 1, 2, 1\rangle|} \\
&= \frac{d-4}{\sqrt{6}} \\
d &= 4 + 3\sqrt{6}
\end{aligned}$$

Example

Find Π such that $(1, 2, 3) \in \Pi$:

$$\Pi \perp \Pi_1 : x - y = 2$$
$$\Pi \perp \Pi_2 : 2x + z = 4$$

These two lines can describe two vectors parallel to Π. Therefore, their cross product is a vector perpendicular to Π, which is the normal of Π.

$$\vec{n} = \vec{n_1} \times \vec{n_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 0 \\ 2 & 0 & 1 \end{vmatrix} = \langle -1, -1, 2\rangle$$

$$\Pi : -1(x-1) - 1(y-2) + 2(z-3) = 0$$

Example

Find Π:

$$l \in \Pi$$

$$l = \Pi_1 \cap \Pi_2$$

$$\Pi_1 : x - 2 = 1$$

$$\Pi_2 : 2x + y + 2 = 4$$

$$\Pi \perp \Pi_3 : x + 2y = 1$$

We can find two points in Π since they must satisfy the following equations:

$$x - z = 1$$
$$2x + y + z = 4$$

We can use $P_0 = (0, 5, -1)$ and $P_1 = (1, 2, 0)$. These two points can describe a vector parallel to Π. We can take the cross product between $\overrightarrow{P_0P_1}$ and $\Pi \perp \Pi_3$ to get the normal of Π.

$$\vec{n} = \overrightarrow{P_0P_1} \times \vec{n_3} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -3 & 1 \\ 1 & 2 & 0 \end{vmatrix} = \langle -2, 1, 5 \rangle$$

$$\Pi : -2(x - 1) + 1(y - 2) + 5(z - 0) = 0$$

Example

Find line l given:

$$P(0, 1, 2) \in l$$
$$l \parallel \Pi : x + y + z = 1$$
$$l \perp l_1 = \begin{cases} x & = 1 + t \\ y & = 1 - t \\ z & = 2t \end{cases}$$

The vector $\vec{a}$ pointing in the direction of l_1 is perpendicular to both Π and l. The normal of Π is perpendicular to l. We can take $\vec{a} \times \vec{n}$ to get a vector $\vec{a_l}$ describing the direction of l.

$$\vec{a_l} = \vec{a_1} \times \vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 2 \\ 1 & 1 & 1 \end{vmatrix} = \langle -3, 1, 2 \rangle$$

$$l = \begin{cases} x & = 0 + 3t \\ y & = 1 + t \\ z & = 2 + 2t \end{cases}$$

Example

Given the parallel lines:

$$l_1 = \begin{cases} x &= 1 + t \\ y &= 0 + 2t \\ z &= 0 + 4t \end{cases} \quad l_2 = \begin{cases} x &= 0 + t \\ y &= 1 + 2t \\ z &= 1 + 4t \end{cases}$$

Find the distance between the lines.
We can pick two arbitrary points on l_1: P_1(1,0,0) and P_2 (2,2,4), and one arbitrary point on l_2: P_3(0,1,1). The area of the parallelogram described by $\overrightarrow{P_1P_3}$ and $\overrightarrow{P_1P_2}$ is:

$$\begin{aligned} \text{area} &= |\overrightarrow{P_1P_3} \times \overrightarrow{P_1P_2}| \\ &= |\overrightarrow{P_1P_3} \cdot dist| \\ dist(l_1, l_2) &= \frac{|\overrightarrow{P_1P_3} \times \overrightarrow{P_1P_2}|}{|\overrightarrow{P_1P_3}|} \\ &= \frac{\left\| \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 4 \\ -1 & 1 & 12 \end{vmatrix} \right\|}{|\langle 1, 2, 4 \rangle|} \\ &= \frac{|\langle -2, -5, 3 \rangle|}{|\langle 1, 2, 4 \rangle|} \\ &= \frac{\sqrt{38}}{\sqrt{21}} \end{aligned}$$

Example

Find the shortest distance between the skew lines:

$$l_1 = \begin{cases} x &= 1 + t \\ y &= 0 + 2t \\ z &= 0 + 4t \end{cases} \quad l_2 = \begin{cases} x &= 0 + s \\ y &= 1 + 2s \\ z &= 1 + s \end{cases}$$

If we take a vector $\vec{a_1}$ in the direction of l_1 and a vector $\vec{a_2}$ in the direction of l_2, we can take their cross product to get a vector describing the normal of a plane Π parallel to both lines such that $l_1 \in \Pi$.

$$\vec{n} = \vec{a_1} \times \vec{a_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 4 \\ 1 & 2 & 1 \end{vmatrix} = \langle -6, 3, 0 \rangle \parallel \langle -2, 1, 0 \rangle$$

The plane Π can be described by:

$$2(x-1) - 1(y-0) + 0(z-0) = 0$$

$$2x - y = 2$$

This reduces the problem to simply the distance between l_2 and Π.

$$\begin{aligned} dist(l_1, l_2) &= comp_{\langle 2,-1,0\rangle}\langle -1,1,1\rangle \\ &= \frac{\langle -1,1,1\rangle \cdot \langle 2,-1,0\rangle}{|\langle 2,-1,0\rangle|} \\ &= \frac{-3}{\sqrt{5}} \end{aligned}$$

If you have any questions, comments, or concerns, please contact me at
alvin@omgimanerd.tech