Multivariable and Vector Calculus

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Vectors

$$A(a_1, a_2, a_3)$$

$$B(b_1, b_2, b_3)$$

$$\vec{AB} = \langle b_1 - a_1, b_2 - a_2, b_3 - a_3 \rangle$$

Vector Operations

$$\vec{a} + \vec{b} = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$$

$$\lambda\vec{a} = \langle \lambda a_1, \lambda a_2, \lambda a_3 \rangle$$

$$|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

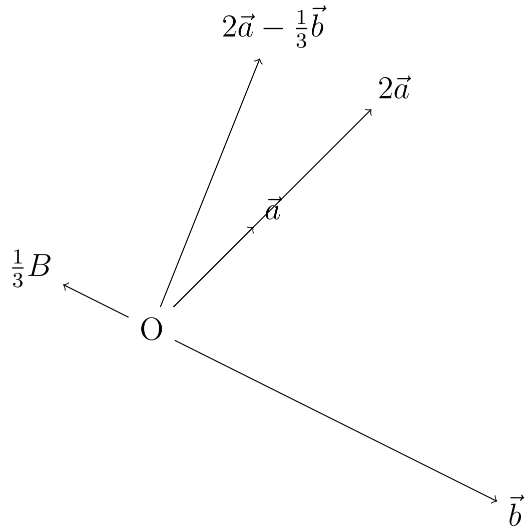
$$|\lambda\vec{a}| = |\lambda||\vec{a}|$$

$$\lambda(\vec{a} + \vec{b}) = \lambda\vec{a} + \lambda\vec{b}$$

$$\vec{a} = \langle a_1, a_2, a_3 \rangle = |\vec{a}|\langle \frac{a_1}{|\vec{a}|} \frac{a_2}{|\vec{a}|} \frac{a_3}{|\vec{a}|} \rangle = |\vec{a}|\langle \cos(\alpha), \cos(\beta), \cos(\gamma) \rangle$$

Example

Draw $2\vec{a} - \frac{1}{3}\vec{b}$:



Example

Given the vector of length 7 which makes an angle of 15° with $\langle 3, 3\rangle$.

$$\vec{a} = 7\langle \cos(30), \cos(60)\rangle = 7\langle \frac{\sqrt{3}}{2}, \frac{1}{2}\rangle = \langle \frac{7\sqrt{3}}{2}, \frac{7}{2}\rangle$$

Alternate Solution:

$$\vec{a} = 7\langle \cos(60), \cos(30)\rangle = 7\langle \frac{1}{2}, \frac{\sqrt{3}}{2}\rangle = \langle \frac{7}{2}, \frac{7\sqrt{3}}{2}\rangle$$

Example

A pilot steers a plane at 500mph towards N 30 E, while the wind blows at 200mph from N 45 W. Find the true course and speed.

$$\begin{aligned}
\vec{t} &= \overrightarrow{pilot} + \overrightarrow{wind} \\
&= 500\langle \cos(60), \cos(30)\rangle + 200\langle \cos(45), -\cos(45)\rangle \\
&= \langle 250, 250\sqrt{3}\rangle + \langle 100\sqrt{2}, -100\sqrt{2}\rangle \\
&= \langle 250 + 100\sqrt{2}, 250\sqrt{3} - 100\sqrt{2}\rangle
\end{aligned}$$

Example

Give a vector of length 3, tangent to $y = 2\sin(x)$ at $\frac{\pi}{6}, 1$.

$$\begin{aligned} y' &= 2\cos(x) \\ y'(\frac{\pi}{6}) &= \sqrt{3} \\ \vec{v} &= \langle 1, \sqrt{3} \rangle \\ |\vec{v}| &= 2 \\ \frac{3\vec{v}}{|\vec{v}|} &= \langle \frac{3}{2}, \frac{\sqrt{3}}{2} \rangle \end{aligned}$$

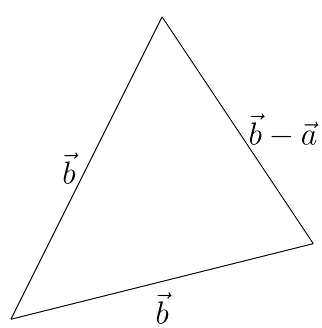
Dot/scalar Product

$$\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_bb_3 = \sum_i^n a_ib_i$$

Properties:

$$\begin{aligned} \vec{a} \cdot \vec{a} &= |\vec{a}|^2 \\ \vec{a} \cdot \vec{b} &= \vec{b} \cdot \vec{a} \\ \vec{a} \cdot (\vec{b} + \vec{c}) &= \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c} \\ (\lambda\vec{a}) \cdot \vec{b} &= \lambda(\vec{a} \cdot \vec{b}) \\ \vec{a} \cdot \vec{b} &= 0 \quad (\vec{a} \perp \vec{b}) \end{aligned}$$

Law of Cosines:



$$|\vec{a} - \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}|\cos(\theta)$$

Using the vector properties:

$$(\vec{b}-\vec{a})\cdot(\vec{b}-\vec{a}) = \vec{a}\cdot\vec{a}+\vec{b}\cdot\vec{b}-2|\vec{a}||\vec{b}|\cos\theta$$
$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$$

Example

$$\vec{a} = \langle 1,2,3\rangle, \vec{b} = \langle 2,x,4\rangle$$

For what x is $\vec{a}\perp\vec{b}$:

$$\begin{aligned}\vec{a}\cdot\vec{b} &= \langle 1,2,3\rangle\cdot\langle 2,x,4\rangle\\ &= 2+2x+12 = 0\\ 2x &= -14\\ x &= -7\end{aligned}$$

Example

Given a triangle ABC where A(1,1,2), B(2,4,3), C(3,5,8), find angle $\angle CAB$. Let $\alpha = \angle CAB$.

$$\begin{aligned}\cos\alpha &= \frac{\overrightarrow{AB}\cdot\overrightarrow{AC}}{|\overrightarrow{AB}||\overrightarrow{AC}|}\\ &= \frac{\langle 1,3,1\rangle\cdot\langle 2,4,6\rangle}{|\langle 1,3,1\rangle||\langle 2,4,6\rangle|}\\ &= \frac{2+12+6}{\sqrt{11}\sqrt{56}}\\ &= \frac{20}{\sqrt{11}\sqrt{56}}\\ \alpha &= \cos^{-1}(\frac{20}{\sqrt{11}\sqrt{56}})\end{aligned}$$

Example

Given two lines:

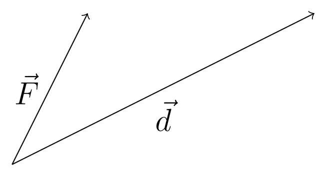
$$l_1 : x-2y = 3$$
$$l_2 : y = 4x+7$$

Find their angle of intersection: We can take two vectors parallel to the the lines and find the angle between them:

$$\langle 2,1\rangle, \langle 1,4\rangle$$

$$\begin{aligned}\cos\theta &= \frac{\langle 2,1\rangle \cdot \langle 1,4\rangle}{|\langle 2,1\rangle||\langle 1,4\rangle|} \\ &= \frac{6}{\sqrt{5}\sqrt{17}} \\ \theta &= \cos^{-1}(\frac{6}{\sqrt{5}\sqrt{17}})\end{aligned}$$

Force and Displacement



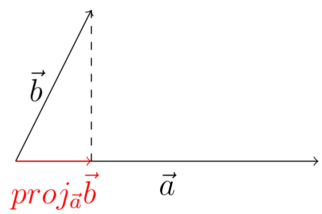
$$W = |\vec{F}|\cos\theta|\vec{d}| = \vec{F} \cdot d$$

Example

A sled is pulled 100ft up a hill with an angle of 30°. The angle between the handle and the sled if 15°. Given that the sled is 20 pounds, what is the amount of work done?

$$\begin{aligned}W &= \vec{F} \cdot \vec{d} \\ &= 20\langle \cos 30, \cos 60\rangle \cdot 100\langle \cos 45, \cos 45\rangle \\ &= 2000\langle \frac{\sqrt{3}}{2}, \frac{1}{2}\rangle \cdot \langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\rangle \\ &= 2000(\frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}) \\ &= 500(\sqrt{6} + \sqrt{2})\end{aligned}$$

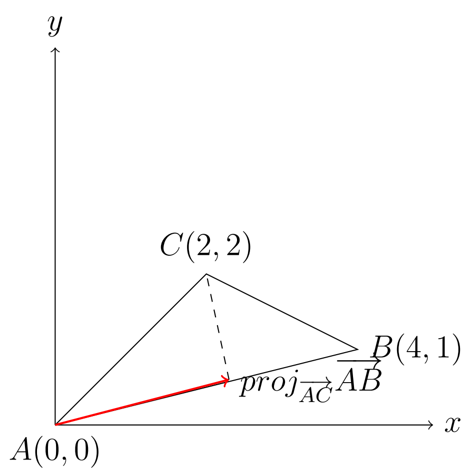
Vector Projections



The scalar project of $\vec{b}$ onto $\vec{a}$ is $comp_{\vec{a}}\vec{b}$.

$$\begin{aligned} comp_{\vec{a}}\vec{b} &= |\vec{b}|\cos\theta \\ &= \frac{|\vec{b}|\cos\theta|\vec{a}|}{|\vec{a}|} \\ &= \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|} \\ proj_{\vec{a}}\vec{b} &= comp_{\vec{a}}\vec{b}\frac{\vec{a}}{|\vec{a}|} \\ &= \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\frac{\vec{a}}{|\vec{a}|} \\ &= \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\vec{a} \end{aligned}$$

Example



$$
\begin{aligned}
comp_{\overrightarrow{AC}}\overrightarrow{AB} &= \frac{\overrightarrow{AB}\cdot\overrightarrow{AC}}{|\overrightarrow{AC}|} \\
&= \frac{\langle 2,2\rangle \cdot \langle 4,1\rangle}{\sqrt{17}} \\
&= \frac{10}{\sqrt{17}}
\end{aligned}
$$

Cross Product

$$
\begin{aligned}
\vec{a}\times\vec{b} &= \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1\rangle \\
&= \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} \\
&= i\begin{bmatrix} a_2 & a_3 \\ b_2 & b_3 \end{bmatrix} - j\begin{bmatrix} a_1 & a_3 \\ b_1 & b_3 \end{bmatrix} - k\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}
\end{aligned}
$$

Properties:

$$|\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}|\sin\theta$$

The magnitude of the cross product of $\vec{a}$ and $\vec{b}$ is the area of a parallelogram with sides $\vec{a}$ and $\vec{b}$. Since we know this, the area of a triangle ABC is $\frac{1}{2}|\vec{AB}\times\vec{AC}|$.

$$\vec{a}\times\vec{b} = \vec{0} \iff \vec{a} \parallel \vec{b}$$

If $(\vec{a}\times\vec{b})\cdot\vec{a} = 0$, then $\vec{a}\times\vec{b}$ is perpendicular to the plane described by $\vec{a}$ and $\vec{b}$.

$$(\vec{a}\times\vec{b})\cdot\vec{a} = 0 \to \vec{a}\times\vec{b} \perp \vec{a}$$

Vice versa:

$$(\vec{a}\times\vec{b})\cdot\vec{b} = 0 \to \vec{a}\times\vec{b} \perp \vec{b}$$

Additionally:

$$
\begin{gathered}
\vec{a}\times\vec{b} = -(\vec{b}\times\vec{a}) \\
\vec{a}\times(\vec{b}+\vec{c}) = \vec{a}\times\vec{b} + \vec{a} + \vec{c} \\
(\lambda\vec{a})\times\vec{b} = \lambda(\vec{a}\times\vec{b})
\end{gathered}
$$

Example

Given a parallelepiped described by the vectors $\vec{a}, \vec{b}, \vec{c}$:

$$\begin{aligned} V &= \text{area of base times height} \\ &= |\vec{a} \times \vec{b}|h \\ &= |\vec{a} \times \vec{b}||c| \cos \angle(\vec{c}, \vec{a} \times \vec{b}) \\ &= (\vec{a} \times \vec{b}) \cdot \vec{c} \\ &= \begin{vmatrix} c_x & c_y & c_z \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} \end{aligned}$$

Consider the following four points: A(1,1,1), B(2,1,3), C(1,2,4), D(2,2,1). Are they coplanar? We can check if $\vec{AB} \times \vec{AC} \parallel \vec{DC} \times \vec{DB}$, or we can check if the volume of the parallelepiped described by $\vec{AB}, \vec{AC}, \vec{AD}$ is 0.

$$\begin{vmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 1 & 1 & 0 \end{vmatrix} = -1(-3) + 0(-3) + 2(-1) = -5 \neq 0$$

You can find all my notes at `http://omgimanerd.tech/notes`. If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech