Sequences and Summations

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Sequences and Summations

Let a_n be the number of binary strings of length n with substring "10".

n	a_n
0	0
1	0
2	1
3	4
n	?

Recursive formula for a_n:

$$a_n = 2a_{n-1} + n - 1$$

Sequences

A **sequence** (called a_n or $f(n)$) is a function from $\mathbb{N}$ to $\mathbb{R}$. Example:

1. $\{a_n\} = \{2, 5, 8, 11, 14, 19, \ldots\}$
2. $\{b_n\} = \{2, 4, 8, 16, 32, 64, \ldots\}$
3. $\{c_n\} = \{1, 2, 5, 10, 17, 26, \ldots\}$

A **linear sequence** or **arithmetic sequence** has a constant difference d between adjacent terms.

$$\{a_0, a_0 + d, a_0 + 2d, a_0 + 3d, \ldots\}$$

$$a_n = dn + a_0$$

Difference sequences:

$$\begin{aligned}
\{a'_n\} &= \{a_n - a_{n-1} \mid n \geq 1\} \\
\{a_n\} &= \{2, 5, 8, 11, \ldots\} \\
\{a'_n\} &= \{5-2, 8-5, 11-8, \ldots\} = \{3, 3, 3, \ldots\} \\
\{c_n\} &= \{1, 2, 5, 10, 17, 26, \ldots\} \\
\{c'_n\} &= \{1, 3, 5, 7, 9, \ldots\} \\
\{c''_n\} &= \{2, 2, 2, 2, 2, \ldots\}
\end{aligned}$$

A **geometric sequence** has a constant ratio r, equal to $\frac{a_{n+1}}{a_n}$.

$$\{a_0, a_0 r, a_0 r^2, a_0 r^3\}$$

$$a_n = a_0 r^n$$

$$\begin{aligned}
\{b_n\} &= \{2, 4, 8, 16, \ldots\} \\
&= 2^n \\
\{b'_n\} &= \{2, 4, 8, 16, \ldots\} \\
\{b''_n\} &= \{2, 4, 8, 16, \ldots\}
\end{aligned}$$

Example

$$\begin{aligned}
\{5, 15, 45, 135, \ldots\} &= \{a_n\} \\
r = \frac{20}{5} &= \frac{80}{4} = 4 \\
a_n &= 5 \times 3^n
\end{aligned}$$

Example

Find a non-recursive formula for $a_n = 2a_{n-1} + n - 1$.

$$\begin{aligned}
a_n &= A2^n + Bn + C \\
a_{n-1} &= A2^{n-1} + B(n-1) + C \\
a_n &= 2(a_{n-1}) + n - 1 \\
A2^n + Bn + C &= 2(A2^{n-1} + B(n-1) + C) + n - 1 \\
B &= -1 = C \\
A &= 1 \\
a_n &= 2^n - n - 1
\end{aligned}$$

Summation Notation

$$\sum_{i=M}^{N} f(i) = f(M) + f(M+1) + \cdots + f(N)$$

$$\sum_{i=M}^{N} f(i) = \sum_{j=M}^{N} f(j) = \sum_{k=M}^{N} f(k)$$

Example

$$\sum_{i=1}^{5} i^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55$$

$$\sum_{j=5}^{8} (\frac{1}{j}) = \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}$$

$$\sum_{k=0}^{5} (2^k) = 2^0 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5 = 2^6 - 1$$

Summation Properties

1. $$\sum_{i=M}^{N} (kf(i)) = k(\sum_{i=M}^{N} f(i))$$

2. $$\sum_{i=M}^{N} (f(i) + g(i)) = \sum_{i=M}^{N} f(i) + \sum_{i=M}^{N} g(i)$$

3. $$\sum_{i=M}^{N} f(i) = \sum_{i=1}^{N} f(i) - \sum_{i=1}^{M-1} f(i)$$

Summation Formulas

1. $$\sum_{i=1}^{n} 1 = n$$

2. Gauss's Formula

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

3.

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

4.

$$\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Geometric Sequence

$$\{a, ar, ar^2, ar^3, \dots\}$$

$$\sum_{i=0}^{n} (ar^i) = a\left(\frac{1-r^{n+1}}{1-r}\right)$$

You can find all my notes at `http://omgimanerd.tech/notes`. If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech