Section 7.4

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Exercise 7

$$\int \frac{x^4}{x-1}\,\mathrm{d}x$$

$$\begin{array}{r|l} & x^3 + x^2 + x + 1 \\ \hline x-1 & x^4 \\ & -x^4 + x^3 \\ \hline & x^3 \\ & -x^3 + x^2 \\ \hline & x^2 \\ & -x^2 + x \\ \hline & x \\ & -x + 1 \\ \hline & 1 \end{array}$$

$$\int \frac{x^4}{x-1}\,\mathrm{d}x = \int x^3 + x^2 + x + 1 + \frac{1}{x-1}\,\mathrm{d}x$$
$$= \frac{x^4}{4} + \frac{x^3}{3} + \frac{x^2}{2} + x + \ln|x+1| + C$$

Exercise 11

$$\int_0^1 \frac{2}{2x^2+3x+1}\,\mathrm{d}x$$

$$\frac{2}{2x^2+3x+1} = \frac{2}{(2x+1)(x+1)} = \frac{A}{2x+1} + \frac{B}{x+1}$$
$$(2x+1)(x+1)\frac{2}{(2x+1)(x+1)} = \frac{A}{2x+1} + \frac{B}{x+1}(2x+1)(x+1)$$
$$2 = A(x+1) + B(2x+1)$$
$$2 = Ax + A + 2Bx + B = (A+2B)x + (A+B)$$

$$A + 2B = 0 \quad A + B = 2$$
$$A + 2(2 - A) = A + 4 - 2A = 0$$
$$A = 4 \quad B = -2$$
$$\frac{2}{2x^2 + 3x + 1} = \frac{4}{2x + 1} - \frac{2}{x + 1}$$

Solve it as an indefinite integral:

$$\int \frac{2}{2x^2 + 3x + 1}\ \mathrm{d}x = \int \frac{4}{2x + 1} - \frac{2}{x + 1}\ \mathrm{d}x$$
$$2\int \frac{2}{2x + 1}\ \mathrm{d}x - 2\int \frac{1}{x + 1}\ \mathrm{d}x$$
$$2\int \frac{2}{2x + 1}\ \mathrm{d}x - 2\int \frac{1}{x + 1}\ \mathrm{d}x$$
$$2\ln|2x + 1| - 2\ln|x + 1| + C$$

Now we apply the original limits of integration:

$$\Big[2\ln|2x + 1| - 2\ln|x + 1|\Big]_0^1$$
$$2\ln|2(1) + 1| - 2\ln|1 + 1| - (2\ln|2(0) + 1| - 2\ln|0 + 1|)$$
$$2\ln|3| - 2\ln|2| - (0)$$
$$= \ln|\frac{9}{4}|$$

Exercise 19

$$\int_0^1 \frac{x^2 + x + 1}{(x + 1)^2(x + 2)}\ \mathrm{d}x$$
$$\frac{x^2 + x + 1}{(x + 1)^2(x + 2)} = \frac{Ax + B}{(x + 1)^2} + \frac{D}{x + 2}$$
$$(x + 1)^2(x + 2)\frac{x^2 + x + 1}{(x + 1)^2(x + 2)} = \frac{Ax + B}{(x + 1)^2} + \frac{D}{x + 2}(x + 1)^2(x + 2)$$
$$x^2 + x + 1 = (Ax + B)(x + 2) + (D)(x^2 + 2x + 1)$$
$$x^2 + x + 1 = Ax^2 + 2Ax + Bx + 2B + Dx^2 + 2Dx + D$$
$$x^2 + x + 1 = (A + D)x^2 + (2A + B + 2D)x + (2B + D)$$
$$A + D = 1 \quad 2A + B + 2D = 1 \quad 2B + D = 1$$
$$2(1 - D) + \frac{1 - D}{2} + 2D = 1$$
$$4 - 4D + 1 - D + 4D = 2$$

$$A = -2 \quad B = -1 \quad D = 3$$

$$\frac{x^2+x+1}{(x+1)^2(x+2)} = \frac{-2x-1}{(x+1)^2} + \frac{3}{x+2}$$

Solve it as an indefinite integral:

$$\int \frac{-2x-1}{(x+1)^2} + \frac{3}{x+2} \, \mathrm{d}x$$

$$-\int \frac{2x+1}{(x+1)^2} \, \mathrm{d}x + \int \frac{3}{x+2} \, \mathrm{d}x$$

$$3\ln|x+2| - \int \frac{2x+1}{(x+1)^2} \, \mathrm{d}x$$

$$Let: u = x+1 \quad x = u-1 \quad \mathrm{d}u = \mathrm{d}x$$

$$3\ln|x+2| - \int \frac{2(u-1)+1}{u^2} \, \mathrm{d}u$$

$$3\ln|x+2| - \int \frac{2u-1}{u^2} \, \mathrm{d}u$$

$$3\ln|x+2| - \int \frac{2}{u} \, \mathrm{d}u - \int u^{-2} \, \mathrm{d}u$$

$$3\ln|x+2| - 2\ln|u| + \frac{1}{u} + C$$

$$3\ln|x+2| - 2\ln|x+1| + \frac{1}{x+1} + C$$

Now we apply the original limits of integration:

$$\left[3\ln|x+2| - 2\ln|x+1| + \frac{1}{x+1}\right]_0^1$$

$$3\ln|1+2| - 2\ln|1+1| + \frac{1}{1+1} - (3\ln|0+2| - 2\ln|0+1| + \frac{1}{0+1})$$

$$3\ln|3| - 2\ln|2| + \frac{1}{2} - (3\ln|2| - 0 + 1)$$

$$3\ln|3| - 2\ln|2| - 3\ln|2| - \frac{1}{2}$$

Exercise 23

$$\int \frac{10}{(x-1)(x^2+9)}$$

$$\frac{10}{(x-1)(x^2+9)} = \frac{A}{x-1} + \frac{Bx+D}{x^2+9}$$

$$(x-1)(x^2+9)\frac{10}{(x-1)(x^2+9)} = \frac{A}{x-1} + \frac{Bx+D}{x^2+9}(x-1)(x^2+9)$$

$$10 = A(x^2+9) + (Bx+D)(x-1)$$

$$10 = Ax^2 + 9A + Bx^2 - Bx + Dx - D$$

$$10 = (A+B)x^2 + (D-B)x + (9A-D)$$

$$A+B=0 \quad D-B=0 \quad 9A-D=10$$

$$D = 9A-10 \quad 9A-10-B=0 \quad 9A-10-(-A)=10$$

$$10A = 20$$

$$A=2 \quad B=-2 \quad D=8$$

$$\frac{10}{(x-1)(x^2+9)} = \frac{2}{x-1} + \frac{-2x+8}{x^2+9}$$

$$\int \frac{10}{(x-1)(x^2+9)}\ \mathrm{d}x = \int \frac{2}{x-1}\ \mathrm{d}x - \int \frac{2x-8}{x^2+9}\ \mathrm{d}x$$

$$2\int \frac{1}{x-1}\ \mathrm{d}x - \int \frac{2x}{x^2+9}\ \mathrm{d}x + \int \frac{8}{x^2+9}\ \mathrm{d}x$$

$$2\ln|x-1| - \ln|x^2+9| + \frac{8}{3}\int \frac{3}{x^2+9}\ \mathrm{d}x$$

$$2\ln|x-1| - \ln|x^2+9| + \frac{8}{3}\tan^{-1}(\frac{x}{3}) + C$$

Exercise 39

$$\int \frac{\sqrt{x+1}}{x}\ \mathrm{d}x$$

$$Let : u = \sqrt{x+1}$$

$$x = u^2 - 1$$

$$\mathrm{d}u = \frac{1}{2\sqrt{x+1}}\ \mathrm{d}x$$

$$2\ \mathrm{d}u\sqrt{x+1} = \ \mathrm{d}x$$

$$2u\ \mathrm{d}u = \ \mathrm{d}x$$

$$\int \frac{u}{u^2-1} 2u\ \mathrm{d}u$$

$$2\int \frac{u^2}{u^2-1}\ \mathrm{d}u$$

$$2\int \frac{u^2-1+1}{u^2-1}\ \mathrm{d}u$$

$$2\int \frac{u^2-1}{u^2-1}+\frac{1}{u^2-1}\ \mathrm{d}u$$

$$2\int 1\ \mathrm{d}u+2\int \frac{1}{(u-1)(u+1)}\ \mathrm{d}u$$

$$2u+C+2\int \frac{1}{(u-1)(u+1)}\ \mathrm{d}u$$

$$\frac{1}{(u-1)(u+1)}=\frac{A}{u-1}+\frac{B}{u+1}$$

$$(u-1)(u+1)\frac{1}{(u-1)(u+1)}=\frac{A}{u-1}+\frac{B}{u+1}(u-1)(u+1)$$

$$1=A(u+1)+B(u-1)$$

$$1=Au+A+Bu-B$$

$$1=(A+B)u+(A-B)$$

$$A+B=0 \quad A-B=1$$

$$(1+B)+B=0 \quad B=-\frac{1}{2} \quad A=\frac{1}{2}$$

$$\frac{1}{(u-1)(u+1)}=\frac{1}{2(u-1)}-\frac{1}{2(u+1)}$$

$$\int \frac{1}{(u-1)(u+1)}\ \mathrm{d}u=\int \frac{1}{2(u-1)}-\frac{1}{2(u+1)}\ \mathrm{d}u$$

$$\frac{1}{2}\int \frac{1}{u-1}-\frac{1}{u+1}\ \mathrm{d}u$$

$$\frac{1}{2}(\ln|u-1|-\ln|u+1|)+C$$

$$2u+C+2\int \frac{1}{(u-1)(u+1)}\ \mathrm{d}u=2u+2(\frac{1}{2}(\ln|u-1|-\ln|u+1|))+C$$

$$2u+\ln|u-1|-\ln|u+1|+C$$

If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech