

Section 7.3

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Calculus II: August 2016 - December 2016

Exercise 9

$$\int_2^3 \frac{dx}{(x^2 - 1)^{\frac{3}{2}}}$$

$$\text{Let : } x = \sec(t)$$

$$dx = \tan(t) \sec(t) dt$$

Solve as an indefinite integral:

$$\int \frac{1}{(\sec^2(t) - 1)^{\frac{3}{2}}} \tan(t) \sec(t) dt$$

$$\int \frac{\tan(t) \sec(t)}{(\tan^2(t))^{\frac{3}{2}}} dt$$

$$\int \frac{\tan(t) \sec(t)}{\tan^3(t)} dt$$

$$\int \frac{\sec(t)}{\tan^2(t)} dt$$

$$\int \frac{1}{\cos(t)} \frac{\cos^2(t)}{\sin^2(t)} dt$$

$$\int \cos(t) \sin^{-2}(t) dt$$

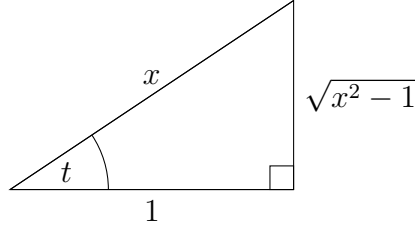
$$f(g(x)) = f'(g(x))g'(x)$$

$$f'(x) = x^{-2} \quad g(x) = \sin(t) \quad g'(x) = \cos(t)$$

$$\text{Therefore : } f(x) = \frac{-1}{x} \quad f(g(x)) = \frac{-1}{\sin(t)}$$

$$\int \cos(t) \sin^{-2}(t) dt = \frac{-1}{\sin(t)} + C$$

Recall that we substituted $x = \sec(t)$, which can be rewritten as $\sec(t) = \frac{x}{1} = \frac{\text{hyp}}{\text{adj}}$. If we imagine a triangle where the hypotenuse is x and the adjacent side is 1, then the opposite side must be $\sqrt{x^2 - 1}$.



Therefore: $\sin(t) = \frac{\text{opp}}{\text{hyp}} = \frac{\sqrt{x^2-1}}{x}$

$$\frac{-1}{\sin(t)} + C = \frac{-x}{\sqrt{x^2-1}} + C$$

Now we integrate using the original limits:

$$\begin{aligned} & \left[\frac{-x}{\sqrt{x^2-1}} \right]_2^3 \\ & \left(\frac{-3}{\sqrt{9-1}} \right) - \left(\frac{-2}{\sqrt{4-1}} \right) \\ & \frac{-3}{2\sqrt{2}} + \frac{2}{\sqrt{3}} \\ & = \frac{2}{\sqrt{3}} - \frac{3}{2\sqrt{2}} \end{aligned}$$

Exercise 19

$$\int \frac{\sqrt{1+x^2}}{x} dx$$

$$\text{Let : } x = \tan(t)$$

$$dx = \sec^2(t) dt$$

$$\int \frac{\sqrt{1+\tan^2(t)}}{\tan(t)} \sec^2(t) dt$$

$$\int \frac{\sqrt{\sec^2(t)}}{\tan(t)} \sec^2(t) dt$$

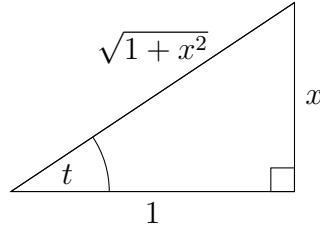
$$\int \frac{\sec(t)}{\tan(t)} (\tan^2(t) + 1) dt$$

$$\int \sec(t) \tan(t) dt + \int \frac{\sec(t)}{\tan(t)} dt$$

$$\int \sec(t) \tan(t) dt + \int \csc(t) dt$$

$$\sec^2(t) + \ln | \csc(t) - \cot(x) | + C$$

Recall that we substituted $x = \tan(t)$, which can be rewritten as $\tan(t) = \frac{x}{1} = \frac{\text{opp}}{\text{adj}}$. If we imagine a triangle where the opposite side is x and the adjacent side is 1 , then the hypotenuse must be $\sqrt{1+x^2}$.



Therefore:

$$\sec(t) = \frac{\text{hyp}}{\text{adj}} = \sqrt{1+x^2}$$

$$\csc(t) = \frac{\text{hyp}}{\text{opp}} = \frac{\sqrt{1+x^2}}{x}$$

$$\cot(t) = \frac{\text{adj}}{\text{opp}} = \frac{1}{x}$$

$$\begin{aligned} \sec^2(t) + \ln |\csc(t) - \cot(x)| + C &= (\sqrt{1+x^2})^2 + \ln \left| \frac{\sqrt{1+x^2}}{x} - \frac{1}{x} \right| + C \\ &= 1+x^2 + \ln \left| \frac{\sqrt{1+x^2}-1}{x} \right| + C \end{aligned}$$

Exercise 21

$$\int_0^{0.6} \frac{x^2}{\sqrt{9-25x^2}} dx$$

$$\text{Let } x = \frac{3}{5} \sin(t)$$

$$dx = \frac{3}{5} \cos(t) dt$$

Solve as an indefinite integral:

$$\int \frac{\frac{9}{25} \sin^2(t)}{\sqrt{9-25\left(\frac{3}{5} \sin(t)\right)^2}} \frac{3}{5} \cos(t) dt$$

$$\frac{27}{125} \int \frac{\sin^2(t)}{\sqrt{9}\sqrt{1-\sin^2(t)}} \cos(t) dt$$

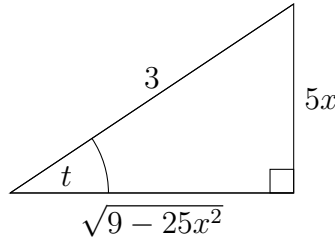
$$\frac{9}{125} \int \frac{\sin^2(t)}{\sqrt{\cos^2(t)}} \cos(t) dt$$

$$\frac{9}{125} \int \sin^2(t) dt$$

$$\frac{9}{125} \int \frac{1-\cos(2t)}{2} dt$$

$$\begin{aligned}
& \frac{9}{250} \int 1 - \cos(2t) \, dt \\
& \frac{9}{250} \left(\int 1 \, dt - \int \cos(2t) \, dt \right) \\
& \frac{9}{250} \left(t - \frac{1}{2} \sin(2t) \right) + C \\
& \frac{9}{250} \left(t - \frac{1}{2} 2 \sin(t) \cos(t) \right) + C \\
& \frac{9}{250} \left(t - \sin(t) \cos(t) \right) + C
\end{aligned}$$

Recall that we substituted $x = \frac{3}{5} \sin(t)$, which can be rewritten as $\sin(t) = \frac{5x}{3} = \frac{\text{opp}}{\text{hyp}}$. If we imagine a triangle where the opposite side is $5x$ and the hypotenuse is 3 , then the adjacent side must be $\sqrt{9 - 25x^2}$.



Therefore:

$$\begin{aligned}
\cos(t) &= \frac{\sqrt{9 - 25x^2}}{3} \\
t &= \sin^{-1}\left(\frac{5x}{3}\right) \\
\frac{9}{250} \left(t - \sin(t) \cos(t) \right) + C &= \frac{9}{250} \left(\sin^{-1}\left(\frac{5x}{3}\right) - \frac{5x}{3} \frac{\sqrt{9 - 25x^2}}{3} \right) + C \\
&= \frac{9 \sin^{-1}\left(\frac{5x}{3}\right)}{250} - \frac{5x \sqrt{9 - 25x^2}}{250} + C
\end{aligned}$$

Now we integrate using the original limits:

$$\begin{aligned}
& \left[\frac{9 \sin^{-1}\left(\frac{5x}{3}\right)}{250} - \frac{5x \sqrt{9 - 25x^2}}{250} \right]_0^{0.6} \\
& \left[\frac{9 \sin^{-1}\left(\frac{3}{3}\right)}{250} - \frac{3\sqrt{9 - 9}}{250} \right] - \left[\frac{9 \sin^{-1}\left(\frac{0}{3}\right)}{250} - \frac{0\sqrt{9 - 0}}{250} \right] \\
& \left[\frac{9\frac{\pi}{2}}{250} - 0 \right] - 0 \\
& = \frac{9\pi}{500}
\end{aligned}$$

Exercise 23

$$\begin{aligned} & \int \frac{dx}{x^2 + 2x + 5} \\ & \int \frac{dx}{x^2 + 2x + 1 + 4} \\ & \int \frac{dx}{(x + 1)^2 + 4} \\ \text{Let : } & x + 1 = 2 \tan(t) \\ & dx = 2 \sec^2(t) dt \\ & \int \frac{2 \sec^2(t)}{(2 \tan(t))^2 + 4} dt \\ & \int \frac{2 \sec^2(t)}{4 \tan^2(t) + 4} dt \\ & \frac{1}{2} \int \frac{\sec^2(t)}{\tan^2(t) + 1} dt \\ & \frac{1}{2} \int \frac{\sec^2(t)}{\sec^2(t)} dt \\ & \frac{1}{2} \int 1 dt \\ & \frac{1}{2} t + C \end{aligned}$$

Given our original substitution $x + 1 = 2 \tan(t)$, we can rewrite this as $\tan(t) = \frac{x+1}{2}$ and get the equation $\tan^{-1}\left(\frac{x+1}{2}\right) = t$. Therefore:

$$\frac{1}{2} t + C = \frac{\tan^{-1}\left(\frac{x+1}{2}\right)}{2} + C$$

If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech