# Section 7.3

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## Exercise 9

$$\int_2^3 \frac{\mathrm{d}x}{(x^2-1)^{\frac{3}{2}}}$$

$$Let : x = \sec(t)$$

$$\mathrm{d}x = \tan(t)\sec(t)\ \mathrm{d}t$$

Solve as an indefinite integral:

$$\int \frac{1}{(\sec^2(t)-1)^{\frac{3}{2}}} \tan(t)\sec(t)\ \mathrm{d}t$$

$$\int \frac{\tan(t)\sec(t)}{(\tan^2(t))^{\frac{3}{2}}}\ \mathrm{d}t$$

$$\int \frac{\tan(t)\sec(t)}{\tan^3(t)}\ \mathrm{d}t$$

$$\int \frac{\sec(t)}{\tan^2(t)}\ \mathrm{d}t$$

$$\int \frac{1}{\cos(t)}\frac{\cos^2(t)}{\sin^2(t)}\ \mathrm{d}t$$

$$\int \cos(t)\sin^{-2}(t)\ \mathrm{d}t$$

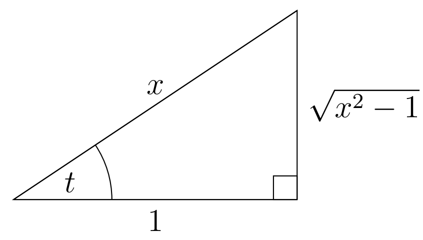
$$f(g(x)) = f'(g(x))g'(x)$$

$$f'(x) = x^{-2} \quad g(x) = \sin(t) \quad g'(x) = \cos(t)$$

$$Therefore : f(x) = \frac{-1}{x} \quad f(g(x)) = \frac{-1}{\sin(t)}$$

$$\int \cos(t)\sin^{-2}(t)\ \mathrm{d}t = \frac{-1}{\sin(t)} + C$$

Recall that we substituted $x = \sec(t)$, which can be rewritten as $\sec(t) = \frac{x}{1} = \frac{hyp}{adj}$. If we imagine a triangle where the hypotenuse is $x$ and the adjacent side is 1, then the opposite side must be $\sqrt{x^2-1}$.




Therefore: $\sin(t) = \frac{opp}{hyp} = \frac{\sqrt{x^2-1}}{x}$

$$\frac{-1}{\sin(t)} + C = \frac{-x}{\sqrt{x^2-1}} + C$$

Now we integrate using the original limits:

$$\left[\frac{-x}{\sqrt{x^2-1}}\right]_2^3$$

$$(\frac{-3}{\sqrt{9-1}}) - (\frac{-2}{\sqrt{4-1}})$$

$$\frac{-3}{2\sqrt{2}} + \frac{2}{\sqrt{3}}$$

$$= \frac{2}{\sqrt{3}} - \frac{3}{2\sqrt{2}}$$

# Exercise 19

$$\int \frac{\sqrt{1+x^2}}{x}\ \mathrm{d}x$$

$$Let : x = tan(t)$$

$$\mathrm{d}x = \sec^2(t)\ \mathrm{d}t$$

$$\int \frac{\sqrt{1+\tan^2(t)}}{\tan(t)} \sec^2(t)\ \mathrm{d}t$$

$$\int \frac{\sqrt{\sec^2(t)}}{\tan(t)} \sec^2(t)\ \mathrm{d}t$$

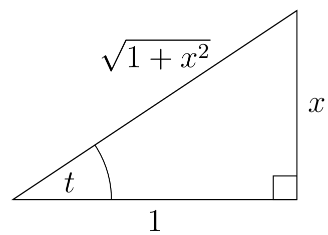
$$\int \frac{\sec(t)}{\tan(t)} (\tan^2(t)+1)\ \mathrm{d}t$$

$$\int \sec(t)\tan(t)\ \mathrm{d}t + \int \frac{\sec(t)}{\tan(t)}\ \mathrm{d}t$$

$$\int \sec(t)\tan(t)\ \mathrm{d}t + \int \csc(t)\ \mathrm{d}t$$

$$\sec^2(t) + \ln|\csc(t) - \cot(x)| + C$$

Recall that we substituted $x = \tan(t)$, which can be rewritten as $\tan(t) = \frac{x}{1} = \frac{opp}{adj}$. If we imagine a triangle where the opposide side is x and the adjacent side is 1, then the hypotenuse must be $\sqrt{1+x^2}$.




Therefore:

$$\sec(t) = \frac{hyp}{adj} = \sqrt{1+x^2}$$

$$\csc(t) = \frac{hyp}{opp} = \frac{\sqrt{1+x^2}}{x}$$

$$\cot(t) = \frac{adj}{opp} = \frac{1}{x}$$

$$\sec^2(t) + \ln|\csc(t) - \cot(x)| + C = (\sqrt{1+x^2})^2 + \ln\left|\frac{\sqrt{1+x^2}}{x} - \frac{1}{x}\right| + C$$

$$= 1 + x^2 + \ln\left|\frac{\sqrt{1+x^2}-1}{x}\right| + C$$

# Exercise 21

$$\int_0^{0.6} \frac{x^2}{\sqrt{9-25x^2}}\ \mathrm{d}x$$

$$Let : x = \frac{3}{5}\sin(t)$$

$$\mathrm{d}x = \frac{3}{5}\cos(t)\ \mathrm{d}t$$

Solve as an indefinite integral:

$$\int \frac{\frac{9}{25}\sin^2(t)}{\sqrt{9-25(\frac{3}{5}\sin(t))^2}}\frac{3}{5}\cos(t)\ \mathrm{d}t$$

$$\frac{27}{125}\int \frac{\sin^2(t)}{\sqrt{9}\sqrt{1-\sin^2(t)}}\cos(t)\ \mathrm{d}t$$

$$\frac{9}{125}\int \frac{\sin^2(t)}{\sqrt{\cos^2(t)}}\cos(t)\ \mathrm{d}t$$

$$\frac{9}{125}\int \sin^2(t)\ \mathrm{d}t$$

$$\frac{9}{125}\int \frac{1-\cos(2t)}{2}\ \mathrm{d}t$$

$$\frac{9}{250}\int 1-\cos(2t)\ \mathrm{d}t$$

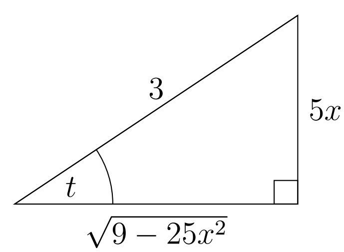
$$\frac{9}{250}\left(\int 1\ \mathrm{d}t-\int \cos(2t)\ \mathrm{d}t\right)$$

$$\frac{9}{250}\left(t-\frac{1}{2}\sin(2t)\right)+C$$

$$\frac{9}{250}\left(t-\frac{1}{2}2\sin(t)\cos(t)\right)+C$$

$$\frac{9}{250}\left(t-\sin(t)\cos(t)\right)+C$$

Recall that we substituted $x=\frac{3}{5}\sin(t)$, which can be rewritten as $\sin(t)=\frac{5x}{3}=\frac{opp}{hyp}$. If we imagine a triangle where the opposide side is $5x$ and the hypotenuse is 3, then the adjacent side must be $\sqrt{9-25x^2}$.




Therefore:

$$\cos(t)=\frac{\sqrt{9-25x^2}}{3}$$

$$t=\sin^{-1}(\frac{5x}{3})$$

$$\frac{9}{250}\left(t-\sin(t)\cos(t)\right)+C=\frac{9}{250}\left(\sin^{-1}(\frac{5x}{3})-\frac{5x}{3}\frac{\sqrt{9-25x^2}}{3}\right)+C$$

$$\frac{9\sin^{-1}(\frac{5x}{3})}{250}-\frac{5x\sqrt{9-25x^2}}{250}+C$$

Now we integrate using the original limits:

$$\left[\frac{9\sin^{-1}(\frac{5x}{3})}{250}-\frac{5x\sqrt{9-25x^2}}{250}\right]_0^{0.6}$$

$$\left[\frac{9\sin^{-1}(\frac{3}{3})}{250}-\frac{3\sqrt{9-9}}{250}\right]-\left[\frac{9\sin^{-1}(\frac{0}{3})}{250}-\frac{0\sqrt{9-0}}{250}\right]$$

$$\left[\frac{9\frac{\pi}{2}}{250}-0\right]-0$$

$$=\frac{9\pi}{500}$$

# Exercise 23

$$\int \frac{\mathrm{d}x}{x^2 + 2x + 5}$$

$$\int \frac{\mathrm{d}x}{x^2 + 2x + 1 + 4}$$

$$\int \frac{\mathrm{d}x}{(x+1)^2 + 4}$$

$$Let : x + 1 = 2\tan(t)$$

$$\mathrm{d}x = 2\sec^2(t)\ \mathrm{d}t$$

$$\int \frac{2\sec^2(t)}{(2\tan(t))^2 + 4}\ \mathrm{d}t$$

$$\int \frac{2\sec^2(t)}{4\tan^2(t) + 4}\ \mathrm{d}t$$

$$\frac{1}{2}\int \frac{\sec^2(t)}{\tan^2(t) + 1}\ \mathrm{d}t$$

$$\frac{1}{2}\int \frac{\sec^2(t)}{\sec^2(t)}\ \mathrm{d}t$$

$$\frac{1}{2}\int 1\ \mathrm{d}t$$

$$\frac{1}{2}t + C$$

Given our original substitution $x + 1 = 2\tan(t)$, we can rewrite this as $\tan(t) = \frac{x+1}{2}$ and get the equation $\tan^{-1}(\frac{x+1}{2}) = t$. Therefore:

$$\frac{1}{2}t + C = \frac{\tan^{-1}(\frac{x+1}{2})}{2} + C$$

If you have any questions, comments, or concerns, please contact me at
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