

The Integral Test and Estimates of Sums

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The Integral Test

If f is continuous, positive, and decreasing, and let $f(n) = a_n$. Then:

1. If $\int_1^{\infty} f(n) \, dn$ converges, then $\sum a_n$ converges.
2. If $\int_1^{\infty} f(n) \, dn$ diverges, then $\sum a_n$ diverges.

Example

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+4}}$$
$$f(x) = \frac{1}{\sqrt{x+4}}$$

$$\begin{aligned}
\int_1^{\infty} f(x) \, dx &= \lim_{b \rightarrow \infty} \int_1^b f(x) \, dx \\
&= \lim_{b \rightarrow \infty} \int_1^b \frac{1}{\sqrt{x+4}} \, dx \\
&= \lim_{b \rightarrow \infty} \int_1^b (x+4)^{-\frac{1}{2}} \, dx \\
&= \lim_{b \rightarrow \infty} \left[\frac{(x+4)^{\frac{1}{2}}}{\frac{1}{2}} \right]_1^b \\
&= \lim_{b \rightarrow \infty} \left[2(b+4)^{\frac{1}{2}} - 2(5)^{\frac{1}{2}} \right] \\
&= \infty \\
&\therefore \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+4}} \text{ diverges}
\end{aligned}$$

Example 2

$$\begin{aligned}
&\sum_{n=1}^{\infty} ne^{-n} \\
&f(x) = xe^{-x} \\
\int_1^{\infty} f(x) \, dx &= \lim_{b \rightarrow \infty} \int_1^b xe^{-x} \, dx \\
&= \lim_{b \rightarrow \infty} \left([-xe^{-x}]_1^b - \int_1^b -e^{-x} \, dx \right) \\
&= \lim_{b \rightarrow \infty} \left[\frac{-b}{e^b} + \frac{1}{e} \right] - \lim_{b \rightarrow \infty} [e^{-b} - e^{-1}] \\
&= 0 + \frac{1}{e} + \frac{1}{e} \\
&\therefore \sum_{n=1}^{\infty} ne^{-n} \text{ converges}
\end{aligned}$$

Example 3

$$\sum_{n=1}^{\infty} \frac{12}{-5^n}$$

$$a_1 + a_2 + a_3 + \dots = \frac{12}{-5^1} + \frac{12}{-5^2} + \frac{12}{-5^3} + \dots$$

$$\frac{a_2}{a_1} = \frac{a_3}{a_2} = r$$

$$r = \frac{\frac{12}{-5^2}}{\frac{12}{-5}} = \frac{-1}{5}; |r| < 1$$

$$\sum_{n=1}^{\infty} \frac{12}{-5^n} = \frac{a}{1-r} = \frac{\frac{12}{-5}}{1-\frac{-1}{5}} = -2$$

Example 4

$$\sum_{n=2}^{\infty} \frac{1}{(1+c)^n} = 2$$

$$\frac{1}{(1+c)^2} + \frac{1}{(1+c)^3} + \dots = 2$$

$$\frac{a}{1-r} = 2$$

$$\frac{\frac{1}{(1+c)^2}}{1-\frac{1}{1+c}} = 2$$

$$\frac{\frac{1}{1+c}}{c} = 2$$

$$\frac{1}{c(1+c)} = 2$$

$$2c + 2c^2 = 1$$

$$2c^2 + 2c - 1 = 0$$

$$c = \frac{-2 \pm \sqrt{4 - 4(2)(-1)}}{4} = \frac{-1 \pm \sqrt{3}}{2}$$

Example 5

$$1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \dots = \sum_{n=1}^{\infty} \frac{1}{2n-1}$$

$$f(x) = \frac{1}{2x-1}$$

$$\begin{aligned} \lim_{b \rightarrow \infty} \int_1^b f(x) \, dx &= \lim_{b \rightarrow \infty} \int_1^b \frac{1}{2x-1} \, dx \\ &= \lim_{b \rightarrow \infty} \left[\frac{\ln(2x-1)}{2} \right]_1^b \\ &= \lim_{b \rightarrow \infty} \left[\frac{\ln(2b-1)}{2} - \frac{\ln(1)}{2} \right] \\ &= \infty \\ &\therefore \sum_{n=1}^{\infty} \frac{1}{2n-1} \text{ diverges} \end{aligned}$$

Problem 21

$$\sum \frac{1}{n \ln(n)}$$

$$f(x) = \frac{1}{x \ln(x)}$$

$$\int_1^{\infty} f(x) \, dx = \lim_{b \rightarrow \infty} \int_1^b \frac{1}{x \ln(x)} \, dx$$

$$\text{Let } \ln(x) = t$$

$$\frac{dt}{dx} = \frac{1}{x} \rightarrow dt = \frac{dx}{x}$$

$$\int \frac{dt}{t} = \ln(t)$$

$$\begin{aligned} \lim_{b \rightarrow \infty} \int_1^b \frac{1}{x \ln(x)} \, dx &= \lim_{b \rightarrow \infty} \left[\ln(\ln(x)) \right]_2^b \\ &= \lim_{b \rightarrow \infty} \left[\ln(\ln(b)) - \ln(\ln(1)) \right] \\ &= \infty \end{aligned}$$

Problem 27

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^p}$$
$$f(x) = \frac{1}{x(\ln(x))^p}$$

$$\begin{aligned} \int_2^{\infty} \frac{1}{x(\ln(x))^p} dx &= \lim_{b \rightarrow \infty} \int_2^b \frac{1}{x(\ln(x))^p} dx \\ &= \lim_{b \rightarrow \infty} \left[\frac{(\ln(b))^{-p+1}}{-p+1} \right]_2^b \\ &= \frac{1}{1-p} \lim_{b \rightarrow \infty} (\ln(b))^{1-p} - \frac{1}{1-p} \ln(\ln(2)) \end{aligned}$$

The series converges if $1 - p < 0$ and diverges if $1 - p \geq 0$.

You can find all my notes at <http://omgimanerd.tech/notes>. If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech