The Integral Test and Estimates of Sums

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Calculus II: August 2016 - December 2016

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The Integral Test

If f is continuous, positive, and decreasing, and let $f(n) = a_n$. Then:

1. If $\int_1^\infty f(n)\ \mathrm{d}n$ converges, then $\sum a_n$ converges.
2. If $\int_1^\infty f(n)\ \mathrm{d}n$ diverges, then $\sum a_n$ diverges.

Example

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+4}}$$

$$f(x) = \frac{1}{\sqrt{x+4}}$$

$$
\begin{aligned}
\int_1^\infty f(x)\ \mathrm{d}x &= \lim_{b\to\infty}\int_1^b f(x)\ \mathrm{d}x \\
&= \lim_{b\to\infty}\int_1^b \frac{1}{\sqrt{x+4}}\ \mathrm{d}x \\
&= \lim_{b\to\infty}\int_1^b (x+4)^{-\frac{1}{2}}\ \mathrm{d}x \\
&= \lim_{b\to\infty}\left[\frac{(x+4)^{\frac{1}{2}}}{\frac{1}{2}}\right]_1^b \\
&= \lim_{b\to\infty}\left[2(b+4)^{\frac{1}{2}} - 2(5)^{\frac{1}{2}}\right] \\
&= \infty \\
&\therefore \sum_{n=1}^{\infty}\frac{1}{\sqrt{n+4}} \quad \text{diverges}
\end{aligned}
$$

Example 2

$$
\sum_{n=1}^{\infty} n\mathrm{e}^{-n}
$$

$$
f(x) = x\mathrm{e}^{-x}
$$

$$
\begin{aligned}
\int_1^\infty f(x)\ \mathrm{d}x &= \lim_{b\to\infty}\int_1^b x\mathrm{e}^{-x}\ \mathrm{d}x \\
&= \lim_{b\to\infty}\left(\Big[-x\mathrm{e}^{-x}\Big]_1^b - \int_1^b -\mathrm{e}^{-x}\ \mathrm{d}x\right) \\
&= \lim_{b\to\infty}\left[\frac{-b}{\mathrm{e}^b} + \frac{1}{\mathrm{e}}\right] - \lim_{b\to\infty}\left[\mathrm{e}^{-b} - \mathrm{e}^{-1}\right] \\
&= 0 + \frac{1}{\mathrm{e}} + \frac{1}{\mathrm{e}} \\
&\therefore \sum_{n=1}^{\infty} n\mathrm{e}^{-n} \quad \text{converges}
\end{aligned}
$$

Example 3

$$
\sum_{n=1}^{\infty}\frac{12}{-5^n}
$$

$$a_1 + a_2 + a_3 + ... = \frac{12}{-5^1} + \frac{12}{-5^2} + \frac{12}{-5^3} + ...$$

$$\frac{a_2}{a_1} = \frac{a_3}{a_2} = r$$

$$r = \frac{\frac{12}{-5^2}}{\frac{12}{-5}} = \frac{-1}{5}; |r| < 1$$

$$\sum_{n=1}^{\infty} \frac{12}{-5^n} = \frac{a}{1-r} = \frac{\frac{12}{-5}}{1 - \frac{-1}{5}} = -2$$

Example 4

$$\sum_{n=2}^{\infty} \frac{1}{(1+c)^n} = 2$$

$$\frac{1}{(1+c)^2} + \frac{1}{(1+c)^3} + ... = 2$$

$$\frac{a}{1-r} = 2$$

$$\frac{\frac{1}{(1+c)^2}}{1 - \frac{1}{1+c}} = 2$$

$$\frac{\frac{1}{1+c}}{c} = 2$$

$$\frac{1}{c(1+c)} = 2$$

$$2c + 2c^2 = 1$$

$$2c^2 + 2c - 1 = 0$$

$$c = \frac{-2 \pm \sqrt{4 - 4(2)(-1)}}{4} = \frac{-1 \pm \sqrt{3}}{2}$$

Example 5

$$1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{2} + ... = \sum_{n=1}^{\infty} \frac{1}{2n-1}$$

$$f(x) = \frac{1}{2n-1}$$

$$\begin{aligned}
\lim_{b\to\infty} \int_1^b f(x)\ \mathrm{d}x &= \lim_{b\to\infty} \int_1^b \frac{1}{2x-1}\ \mathrm{d}x \\
&= \lim_{b\to\infty} \left[\frac{\ln(2x-1)}{2}\right]_1^b \\
&= \lim_{b\to\infty} \left[\frac{\ln(2b-1)}{2} - \frac{\ln(1)}{2}\right] \\
&= \infty \\
&\therefore \sum_{n=1}^{\infty} \frac{1}{2n-1} \quad \text{diverges}
\end{aligned}$$

Problem 21

$$\sum \frac{1}{n\ln(n)}$$

$$f(x) = \frac{1}{x\ln(x)}$$

$$\int_1^{\infty} f(x)\ \mathrm{d}x = \lim_{b\to\infty} \int_1^b \frac{1}{x\ln(x)}\ \mathrm{d}x$$

$$\text{Let} : \ln(x) = t$$

$$\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{1}{x} \to \ \mathrm{d}t = \frac{\mathrm{d}x}{x}$$

$$\int \frac{\mathrm{d}t}{t} = \ln(t)$$

$$\begin{aligned}
\lim_{b\to\infty} \int_1^b \frac{1}{x\ln(x)}\ \mathrm{d}x &= \lim_{b\to\infty} \Big[\ln(\ln(x))\Big]_2^b \\
&= \lim_{b\to\infty} \Big[\ln(\ln(b)) - \ln(\ln(1))\Big] \\
&= \infty
\end{aligned}$$

Problem 27

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^p}$$

$$f(x) = \frac{1}{x(\ln(x))^p}$$

$$\begin{aligned}
\int_2^{\infty} \frac{1}{x(\ln(x))^p}\ \mathrm{d}x &= \lim_{b\to\infty} \int_2^{b} \frac{1}{x(\ln(x))^p}\ \mathrm{d}x \\
&= \lim_{b\to\infty} \left[\frac{(\ln(b))^{-p+1}}{-p+1}\right]_2^b \\
&= \frac{1}{1-p} \lim_{b\to\infty} (\ln(b))^{1-p} - \frac{1}{1-p} \ln(\ln(2))
\end{aligned}$$

The series converges if $1-p<0$ and diverges if $1-p$.

You can find all my notes at `http://omgimanerd.tech/notes`. If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech