Series

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Series

Sequence $\Big\{a_1, ...a_n, ...\Big\} \equiv \Big\{a_n\Big\}$.

$$\sum a_n = a_1 + a_2 + a_3 + ... + a_n + ... = S_n$$

$$\begin{aligned} \text{If}: &\quad \lim_{n\to\infty} S_n = S \\ \text{Then}: &\quad \sum a_n \text{ converges}, \quad \sum a_n = S, \quad \sum a_n < \infty \end{aligned}$$

Geometric Series

$$S_n = a + ar + ar^2 + ar^3 + ...$$
$$rS_n = ar + ar^2 + ar^3 + ...$$
$$S_n(1-r) = a - ar^n$$
$$S_n = \frac{a(1-r^n)}{1-r}$$
$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r}$$
$$\lim_{n\to\infty} S_n = \frac{a}{1-r}; |r| < 1$$

If $|r| < 1$, the series converges, if $|r| \geq 1$, the series diverges. If $\sum a_n$ converges, then $\lim_{n\to\infty} a_n = 0$. Proof:

$$S_n - S_{n-1} = a_n$$
$$\lim_{n\to\infty} S_n - S_{n-1} = \lim_{n\to\infty} a_n$$
$$S - S = \lim_{n\to\infty} a_n = 0$$

Problem 23

$$\sum_{k=2}^{\infty} \frac{k^2}{k^2-1} = \frac{4}{3} + \frac{9}{8} + ...$$

$$a_n = \frac{n^2}{n^2-1}$$

$$\sum a_n = a_1 + a_1 r + a_1 r^2 + ...$$

$$\frac{a_3}{a_2} = \frac{a_2}{a_1} = r$$

$$\sum a_n < \infty$$

$$\lim_{n\to\infty} a_n \to 1 \therefore \sum a_n \to \infty \quad \text{(Diverging)}$$

Problem 25

$$\sum \frac{1+2^n}{2^n}$$

$$\begin{aligned} \sum \frac{1+2^n}{2^n} &= \sum \frac{1}{3^n} + \sum \frac{2^n}{3^n} \\ &= \sum (\frac{1}{3})^n + \sum (\frac{2}{3})^n \\ &= \frac{a}{1-r} + \frac{a}{1-r} \\ &= \frac{\frac{1}{3}}{1-\frac{1}{3}} + \frac{\frac{2}{3}}{1-\frac{2}{3}} \\ &= \frac{1}{2} + \frac{2}{1} \\ &= \frac{5}{2} \end{aligned}$$

Problem 29

$$\sum_{n=1}^{\infty} \ln(\frac{n^2+1}{2n^2+1})$$

$$a_n = \ln(\frac{n^2+1}{2n^2+1})$$

$$\begin{aligned}\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \ln(\frac{n^2+1}{2n^2+1})\\ &= \lim_{n\to\infty} \ln(\frac{n^2(1+\frac{1}{n^2})}{n^2(2+\frac{1}{n^2})})\\ &= \lim_{n\to\infty} \ln(\frac{1+\frac{1}{n^2}}{2+\frac{1}{n^2}})\\ &= \ln(\frac{1}{2})\end{aligned}$$

$$\lim_{n\to\infty} a_n = \ln(\frac{1}{2}) \therefore \sum a_n = \infty \quad \text{(Diverges)}$$

Problem 33

$$\sum_{n=1}^{\infty}\left[\frac{1}{\mathrm{e}^n}+\frac{1}{n(n+1)}\right]$$

$$\sum_{n=1}^{\infty}\left[\frac{1}{\mathrm{e}^n}+\frac{1}{n(n+1)}\right] = \sum_{n=1}^{\infty}\frac{1}{\mathrm{e}^n}+\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$$

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{\mathrm{e}^n} &= \frac{1}{\mathrm{e}}+\frac{1}{\mathrm{e}^2}+\frac{1}{\mathrm{e}^3}+\ldots\\ &= \frac{1}{1-r}\\ &= \frac{\frac{1}{\mathrm{e}}}{1-\frac{1}{\mathrm{e}}}\\ &= \frac{1}{\mathrm{e}-1}\\ \sum_{n=1}^{\infty}\frac{1}{n(n+1)} &= \sum_{n=1}^{\infty}\frac{1}{n}-\frac{1}{n+1}\\ &= (1-\frac{1}{2})+(\frac{1}{2}-\frac{1}{3})+(\frac{1}{3}-\frac{1}{4})+\ldots\\ &= 1\\ \sum_{n=1}^{\infty}\left[\frac{1}{\mathrm{e}^n}+\frac{1}{n(n+1)}\right] &= \frac{1}{\mathrm{e}-1}+1\end{aligned}$$

Problem 35

$$\sum_{n=2}^{\infty} \frac{2}{n^2-1}$$

$$\begin{aligned}
a_n &= \frac{2}{n^2-1} \\
&= \frac{2}{(n-1)(n+1)} \\
&= \frac{(n+1)-(n-1)}{(n-1)(n+1)} \\
&= \frac{1}{n-1} - \frac{1}{n+1} \\
\sum_{n=2}^{\infty} a_n &= \sum_{n=2}^{\infty} \frac{1}{n-1} - \frac{1}{n+1} \\
&= (1-\frac{1}{3}) + (\frac{1}{2}-\frac{1}{4}) + (\frac{1}{3}-\frac{1}{5}) + (\frac{1}{4}-\frac{1}{6}) + ... \\
S_n &= (1-\frac{1}{3}) + (\frac{1}{2}-\frac{1}{4}) + ... + (\frac{1}{n-1}-\frac{1}{n+1}) \\
&= 1 + \frac{1}{2} - \frac{1}{n-1} - \frac{1}{n} \\
\lim_{n\to\infty} S_n &= \frac{3}{2}
\end{aligned}$$

p-series

$$\sum \frac{1}{n^p}$$

If $p > 1$, the series is convergent. If $p \leq 1$, the series is divergent.

- $\sum \frac{1}{n}$ diverges.
- $\sum \frac{1}{n^2}$ converges.
- $\sum \frac{1}{n^{\frac{1}{3}}}$ diverges.
- $\sum \frac{1}{n^{-2}}$ diverges.

You can find all my notes at `http://omgimanerd.tech/notes`. If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech