Areas and Lengths in Polar Coordinates

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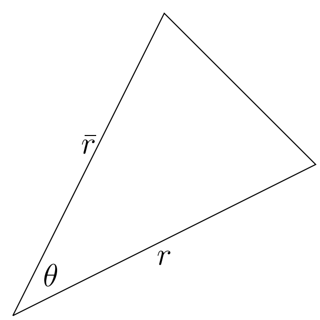
Areas and Lengths in Polar Coordinates

Discrete Area



$$A_i = \frac{1}{2}r^2\theta$$

$$A = \sum_{i=i}^{n} A_i$$

$$= \sum_{i=1}^{n} \frac{1}{2}r^2\Delta\theta$$

$$= \frac{1}{2}\int_a^b r^2 \mathrm{d}\theta$$

Arc Length

$$L = \int_a^b \sqrt{x^2 + y^2}\mathrm{d}\theta$$

$$r = f(\theta) \quad \text{compared to} \quad y = f(x)$$

$$x = r\cos(\theta) = f(\theta)\cos(\theta) \Longrightarrow \frac{\mathrm{d}x}{\mathrm{d}\theta} = -f(\theta)\sin(\theta) + f'(\theta)\cos(\theta)$$

$$y = r\sin(\theta) = f(\theta)\sin(\theta) \Longrightarrow \frac{\mathrm{d}y}{\mathrm{d}\theta} = -f'(\theta)\sin(\theta) + f(\theta)\cos(\theta)$$

$$\begin{aligned}
x^2 + y^2 &= (-r\sin(\theta) + \frac{\mathrm{d}r}{\mathrm{d}\theta}\cos(\theta))^2 + (\frac{\mathrm{d}r}{\mathrm{d}\theta}\sin(\theta) + r\cos(\theta))^2 \\
&= r^2\sin^2(\theta) + (\frac{\mathrm{d}r}{\mathrm{d}\theta})^2\cos^2(\theta) - 2r\frac{\mathrm{d}r}{\mathrm{d}\theta}\sin(\theta)\cos(\theta) + \\
&\quad (\frac{\mathrm{d}r}{\mathrm{d}\theta})^2\sin^2(\theta) + r^2\cos^2(\theta) + 2r\frac{\mathrm{d}r}{\mathrm{d}\theta}\sin(\theta)\cos(\theta) \\
&= r^2(\cos^2(\theta) + \sin^2(\theta)) + (\frac{\mathrm{d}r}{\mathrm{d}\theta})^2(\cos^2(\theta) + \sin^2(\theta)) \\
&= r^2 + (\frac{\mathrm{d}r}{\mathrm{d}\theta})^2 \\
L &= \int_a^b \sqrt{r^2 + (\frac{\mathrm{d}r}{\mathrm{d}\theta})^2}\mathrm{d}\theta
\end{aligned}$$

Example 1

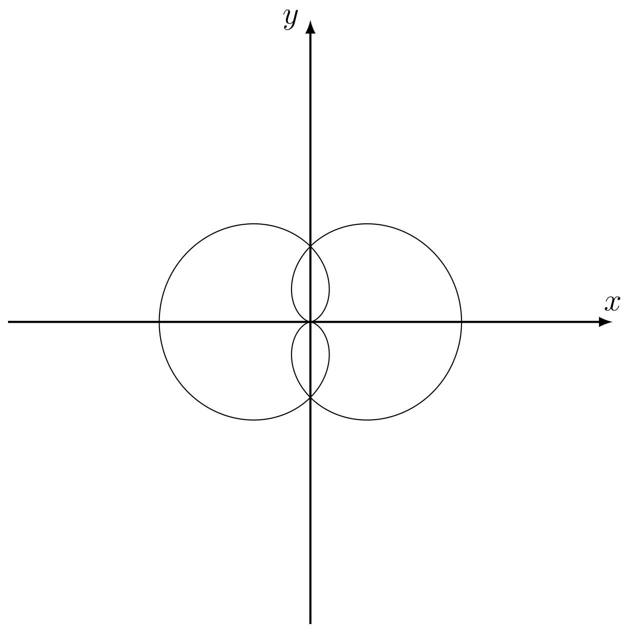
$$r = 2 - \sin(\theta)$$

$$
\begin{aligned}
A &= \frac{1}{2}\int_0^{2\pi} r^2 \mathrm{d}\theta \\
&= \frac{1}{2}\int_0^{2\pi} (2-\sin(\theta))^2 \mathrm{d}\theta \\
&= \frac{1}{2}\int_0^{2\pi} 4+\sin^2(\theta)-4\sin(\theta)\mathrm{d}\theta \\
&= \frac{1}{2}\int_0^{2\pi} 4+(\frac{1-\cos(2\theta)}{2})-4\sin(\theta)\mathrm{d}\theta \\
&= \frac{1}{2}\left[4\theta+\frac{\theta}{2}-\frac{\sin(2\theta)}{4}+4\cos(\theta)\right]_0^{2\pi} \\
&= \frac{1}{2}\left[\frac{9}{2}2\pi-\frac{\sin 4\pi}{4}+4\cos(2\pi)-(-\frac{\sin(\theta)}{4}+4\cos(\theta))\right] \\
&= \frac{1}{2}\left[9\pi+4-4\right] \\
&= \frac{9\pi}{2}
\end{aligned}
$$

Problem 30

Find the integral for the area in common between the two functions:

$$r = 1+\cos(\theta) \quad r = 1-\cos(\theta)$$

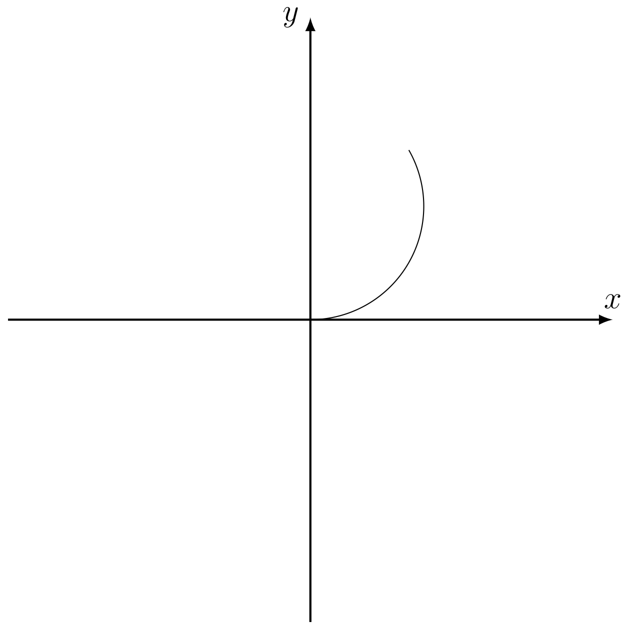


$$A = (4)\frac{1}{2}\int_0^{\frac{\pi}{2}} (1-\cos(\theta))^2 \mathrm{d}\theta$$

Problem 35

Find the arc length:

$$r = 3\sin(\theta) \quad \theta \in \left[0, \frac{\pi}{3}\right]$$



$$
\begin{aligned}
L &= \int_a^b \sqrt{r^2 + (\frac{\mathrm{d}r}{\mathrm{d}\theta})^2}\mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{3}} \sqrt{(3\sin(\theta))^2 + (3\cos(\theta))^2}\mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{3}} \sqrt{9(sin^2(\theta) + \cos^2(\theta))}\mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{3}} 3\mathrm{d}\theta \\
&= \Big[3\theta\Big]_0^{\frac{\pi}{3}} \\
&= \pi
\end{aligned}
$$

Practice Problem 46

$$r = \mathrm{e}^{2\theta} \quad \theta \in \Big[0, 2\pi\Big]$$

$$
\begin{aligned}
L &= \int_a^b \sqrt{r^2 + (\frac{\mathrm{d}r}{\mathrm{d}\theta})^2}\mathrm{d}\theta \\
&= \int_0^{2\pi} \sqrt{(e^{2\theta})^2 + (2\mathrm{e}^{2\theta})^2}\mathrm{d}\theta \\
&= \int_0^{2\pi} \sqrt{\mathrm{e}^{4\theta} + 4\mathrm{e}^{4\theta}}\mathrm{d}\theta \\
&= \int_0^{2\pi} \mathrm{e}^{2\theta}\sqrt{5}\mathrm{d}\theta \\
&= \left[\sqrt{5}\frac{e^{2\theta}}{2}\right]_0^{2\pi} \\
&= \frac{\mathrm{e}^{4\pi}\sqrt{5}}{2} - \frac{\sqrt{5}}{2}
\end{aligned}
$$

You can find all my notes at `http://omgimanerd.tech/notes`. If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech