Polar Coordinates

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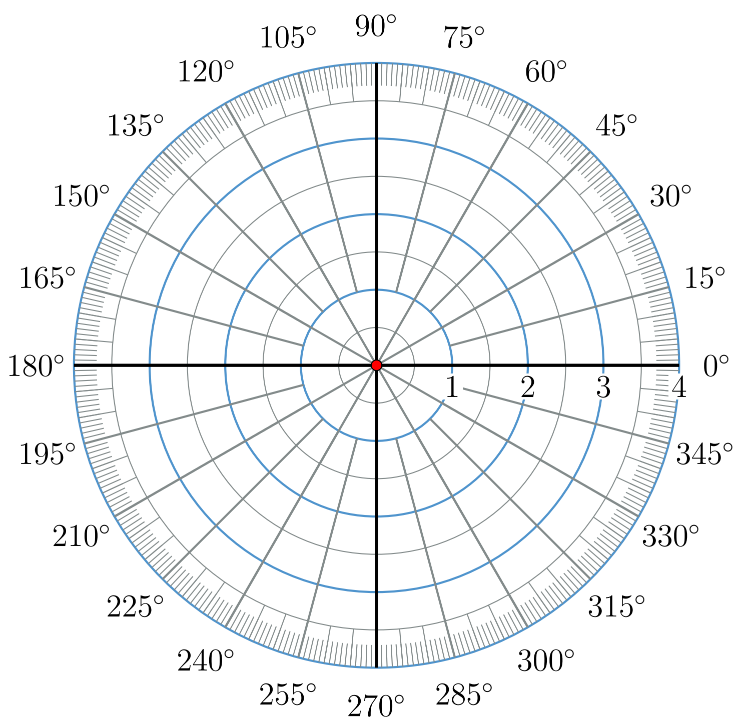
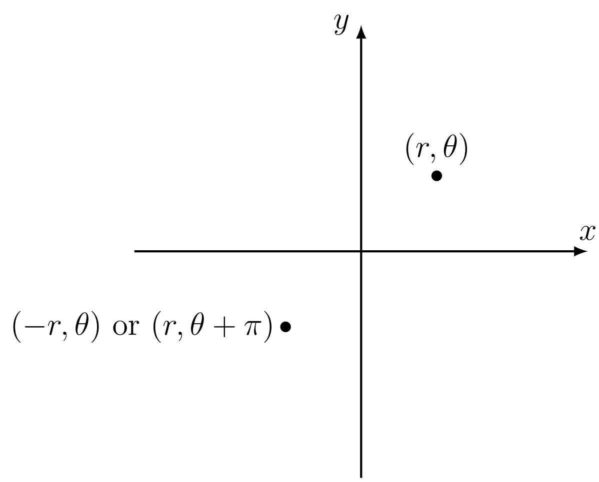
Polar Coordinates

Instead of an x,y coordinate, polar coordinates are defined by an angle and a radius.

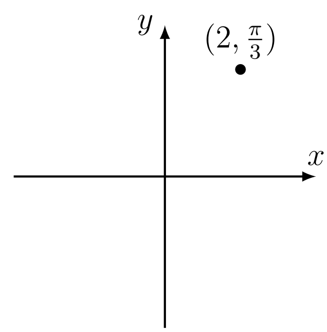


$$x = r\cos(\theta) \quad y = r\sin(\theta)$$



Problem 1a

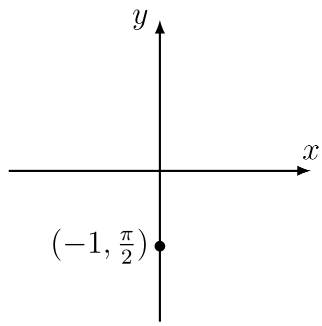
Give two other representations of $(2, \frac{\pi}{3})$.



$$(2, \frac{\pi}{3}) = (-2, \frac{4\pi}{3}) = (2, \frac{7\pi}{3})$$

Problem 1b

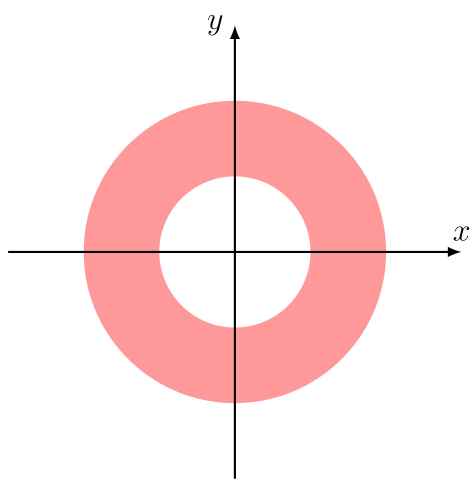
Give two other representations of $(-1, \frac{\pi}{2})$.



$$(-1, \frac{\pi}{2}) = (-1, \frac{5\pi}{2}) = (1, \frac{3\pi}{2})$$

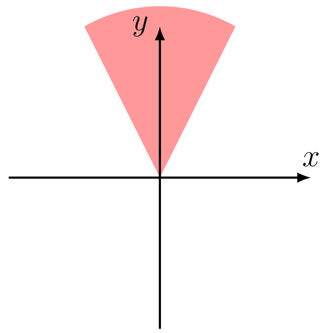
Problem 7

Shade the region $1 \leq r \leq 2$.



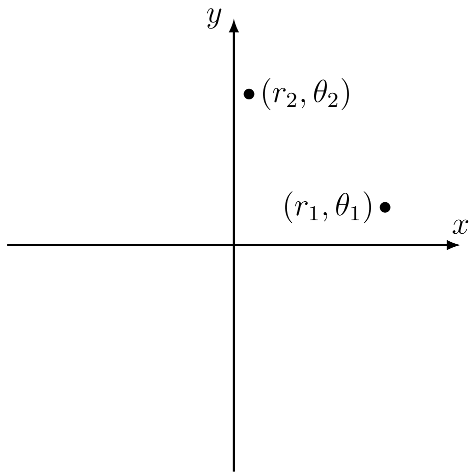
Problem 8

Shade the region $r \geq 0$; $\frac{\pi}{3} \leq 0 \leq \frac{2\pi}{3}$.



Problem 14

Find the distance between the two points:



$$\begin{aligned} P_1(x_1, y_1) &\quad P_2(x_2, y_2) \\ x_1 = r_1\cos(\theta_1) &\quad x_2 = r_2\sin(\theta_2) \\ y_1 = r_1\sin(\theta_1) &\quad y_2 = r_2\sin(\theta_2) \end{aligned}$$

$$\begin{aligned} P_1P_2 &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(r_2\cos(\theta_2) - r_1\cos(\theta_1))^2 + (r_2\sin(\theta_2) - r_1\sin(\theta_1))^2} \\ &= \sqrt{(r_2)^2\cos^2(\theta_2) + (r_1)^2\cos^2(\theta_1) + (r_2)^2\sin^2\theta_2 + (r_1)^2\sin^2(\theta_1) - 2r_1r_2\sin(\theta_1)\sin(\theta_2)} \\ &= \sqrt{(r_1)^2 + (r_2)^2 - 2r_1r_2\cos(\theta_1 - \theta_2)} \\ &\text{(Law of Cosines)} \end{aligned}$$

We could also have drawn a triangle and applied the Law of Cosines, but this shows a the derivation of the law.

Problem 21

Convert the equation to polar coordinates:

$$x = 3$$

$$r\cos(\theta) = 3$$

Problem 22

Convert the equation to polar coordinates:

$$x^2 + y^2 = 9$$

$$\begin{aligned}
(r\cos(\theta))^2 + (r\sin(\theta))^2 &= 9 \\
r^2(\cos^2(\theta) + \sin^2(\theta)) &= 9 \\
r^2 &= 9 \\
r &= 3
\end{aligned}$$

Example 1

Find the slope of the tangent line of $r = 2 - \sin(\theta)$ at the point $\theta = \frac{\pi}{3}$.

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y/\,\mathrm{d}\theta}{\mathrm{d}x/\,\mathrm{d}\theta} \\
x &= r\cos(\theta) \\
&= (2 - \sin(\theta))\cos(\theta) \\
\frac{\mathrm{d}x}{\mathrm{d}\theta} &= (2 - \sin(\theta))(-\sin(\theta)) + (\cos(\theta))(-\cos(\theta)) \\
&= -2\sin(\theta) - \cos(2\theta) \\
y &= r\cos(\theta) \\
&= (2 - \sin(\theta))\cos(\theta) \\
&= 2\sin(\theta) - \sin^2(\theta) \\
\frac{\mathrm{d}y}{\mathrm{d}\theta} &= 2\cos(\theta) - 2\sin(\theta)\cos(\theta) \\
&= 2\cos(\theta) - \sin(2\theta) \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{2\cos(\theta) - \sin(2\theta)}{-2\sin(\theta) - \cos(2\theta)} \\
&= \frac{2\cos(\frac{\pi}{3}) - \sin(\frac{2\pi}{3})}{-2\sin(\frac{\pi}{3}) - \cos(\frac{2\pi}{3})}
\end{aligned}$$

Example 2

Finding the horizontal and vertical tangents:

$$r = 1 + \cos(\theta)$$

Horizontal Tangent:

$$\frac{\mathrm{d}y}{\mathrm{d}\theta} = 0$$

$$\begin{aligned}
y &= r\sin(\theta) \\
y &= (1+\cos(\theta))\sin(\theta) \\
\frac{\mathrm{d}y}{\mathrm{d}\theta} &= -\sin^2(\theta) + (1+\cos(\theta))\cos(\theta) \\
&= \cos(\theta) + \cos^2(\theta) - \sin^2(\theta) \\
&= \cos^2(\theta) - (1-\cos^2(\theta)) + \cos(\theta) = 0
\end{aligned}$$

$$\cos^2(\theta) - (1-\cos^2(\theta)) + \cos(\theta) = 0$$
$$\cos^2(\theta) - 1 + \cos^2(\theta) + \cos(\theta) = 0$$
$$2\cos^2(\theta) + \cos(\theta) + 1 = 0$$
$$2\cos^2(\theta) + 2\cos(\theta) - \cos(\theta) - 1 = 0$$
$$2\cos(\theta)(\cos(\theta)+1) - (\cos(\theta)+1) = 0$$
$$(\cos(\theta)+1)(2\cos(\theta)-1) = 0$$
$$\cos(\theta) = \frac{1}{2} \quad \cos(\theta) = -1$$
$$\theta = \frac{\pi}{3}, \pi, \frac{5\pi}{3}$$

Points of tangency:

$$\begin{aligned}
(r,\theta) &= (\frac{3}{2}, \frac{\pi}{3}) \\
&= (0,\pi) \\
&= (\frac{3}{2}, \frac{5\pi}{3})
\end{aligned}$$

Vertical Tangent:

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = 0$$

$$\begin{aligned}
x &= r\cos(\theta) \\
&= (1+\cos(\theta))\cos(\theta) \\
r &= \cos(\theta) + \cos^2(\theta) \\
\frac{\mathrm{d}x}{\mathrm{d}\theta} &= -\sin(\theta) + 2\cos(\theta)(-2\sin(\theta)) \\
&= -\sin(\theta)(1+2\cos(\theta)) = 0
\end{aligned}$$

$$-\sin(\theta)(1+2\cos(\theta)) = 0$$
$$\sin(\theta) = 0 \quad \cos(\theta) = -\frac{1}{2}$$
$$\theta = 0, \pi, \frac{2\pi}{3}, \frac{4\pi}{3}$$

Points of tangency:

$$\begin{aligned}(r,\theta) &= (\frac{1}{2}, \frac{2\pi}{3}) \\ &= (\frac{1}{2}, \frac{4\pi}{3}) \\ &= (2, 0) \\ &= (0, \pi) \leftarrow \text{Undefined at this point}\end{aligned}$$

You can find all my notes at `http://omgimanerd.tech/notes`. If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech