

Calculus With Parametric Curves

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Calculus With Parametric Curves

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{d}{dx} \left(\frac{dy}{dx} \right) \\ &= \frac{d}{dt} \left(\frac{dy}{dx} \right) \frac{dt}{dx} \\ &= \frac{\frac{d}{dt} \left(\frac{dy}{dx} \right)}{\frac{dx}{dt}}\end{aligned}$$

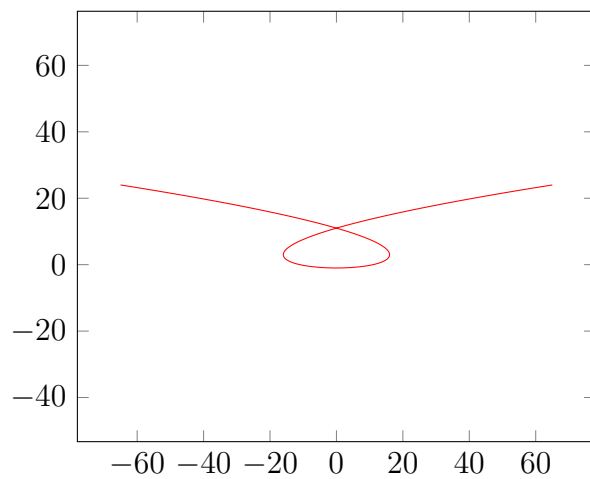
Using this, we can write the arc length function as:

$$\begin{aligned}L &= \int_a^b \sqrt{1 + \left(\frac{dy}{dx} \right)^2} dx \\ &= \int_a^b \sqrt{1 + \left(\frac{dy/dt}{dx/dt} \right)^2} dx \\ &= \int_a^b \frac{\sqrt{(dx/dt)^2 + (dy/dt)^2}}{dx/dt} dx \\ &= \int_a^b \sqrt{\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2} dt\end{aligned}$$

Example 1

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$:

$$x = t^3 - 12t \quad y = t^2 - 1$$



$$\frac{dx}{dt} = 3t^2 - 12 = 3(t^2 - 4)$$

$$\frac{dy}{dt} = 2t$$

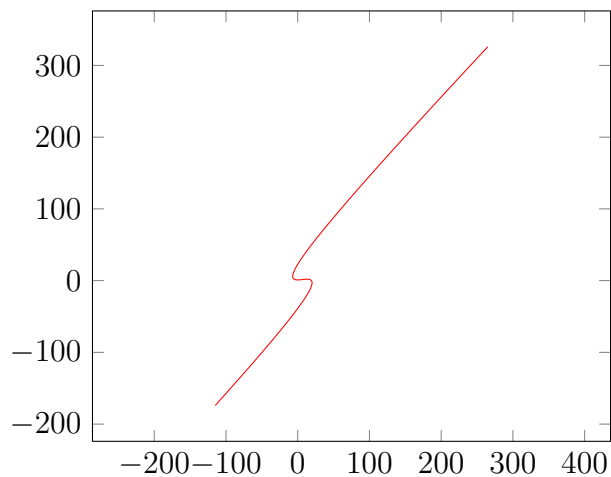
$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t}{3t^2 - 12}$$

$$\begin{aligned}
\frac{d}{dt}\left(\frac{dy}{dx}\right) &= \frac{(3t^2 - 12)(2) - 2t(6t)}{(3t^2 - 12)^2} \\
&= \frac{6t^2 - 24 - 12t^2}{(3t^2 - 12)^2} \\
&= \frac{-6(t^2 + 4)}{9(t^2 - 4)^2} \\
\frac{d^2y}{dx^2} &= \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} \\
&= \frac{2t}{3t^2 - 12} \times \frac{1}{3(t^2 - 4)} \\
&= \frac{-6(t^2 + 4)}{27(t^2 - 4)^3} \\
&= -\frac{2(t^2 + 4)}{9(t^2 - 4)^3}
\end{aligned}$$

Example 2

Find $\frac{dy}{dx}$:

$$x = 2t^3 + 3t^2 - 12t \quad y = 2t^3 + 3t^2 + 1$$



$$\begin{aligned}\frac{dx}{dt} &= 6t^2 + 6t - 12 \\ &= 6(t^2 + t - 2)\end{aligned}$$

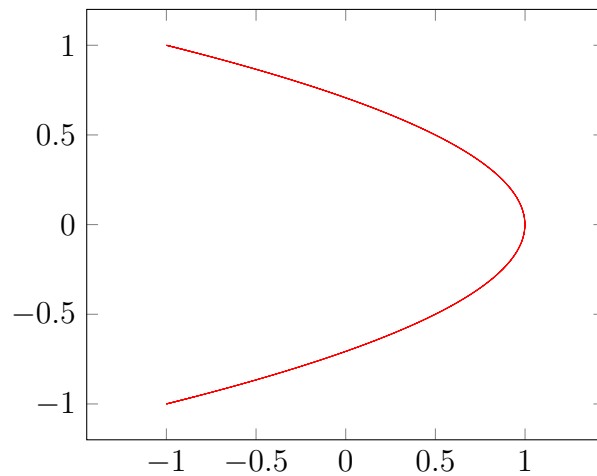
$$\begin{aligned}\frac{dy}{dt} &= 6t^2 + 6t \\ &= 6t(t + 1)\end{aligned}$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \\ &= \frac{6t(t + 1)}{6(t^2 + t - 2)} \\ &= \frac{t(t + 1)}{(t + 2)(t - 2)}\end{aligned}$$

Example 3

Find the points at which the function is convex:

$$x = \cos(2t) \quad y = \cos(t) \quad 0 < t < \pi$$



$$\begin{aligned}
\frac{dx}{dt} &= -2 \sin(2t) \\
\frac{dy}{dt} &= -\sin^2(t) \\
\frac{dy}{dx} &= \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \\
&= \frac{-\sin(t)}{-2 \sin(2t)} \\
&= \frac{\sin(t)}{2 \times 2 \sin(t) \cos(t)} \\
&= \frac{\sec(t)}{4} \\
\frac{d^2y}{dx^2} &= \frac{\frac{d}{dt} \left(\frac{dy}{dx} \right)}{\frac{dx}{dt}} \\
&= \frac{\sec(t) \tan(t)}{4} \times \frac{1}{-2 \sin(2t)} \\
&= -\frac{\sec(t) \tan(t)}{8 \sin(2t)} \\
&= -\frac{\sin(t)}{8 \cos(t) \cos(t) 2 \sin(t) \cos(t)} \\
&= -\frac{\sec^3(t)}{16}
\end{aligned}$$

The function is convex wherever $\frac{d^2y}{dx^2} > 0$:

$$-\frac{\sec^3(t)}{16} > 0$$

$$\sec^3(t) < 0$$

$$\sec(t) < 0$$

$$\cos(t) < 0$$

$$\frac{\pi}{2} < t < \frac{3\pi}{2}$$

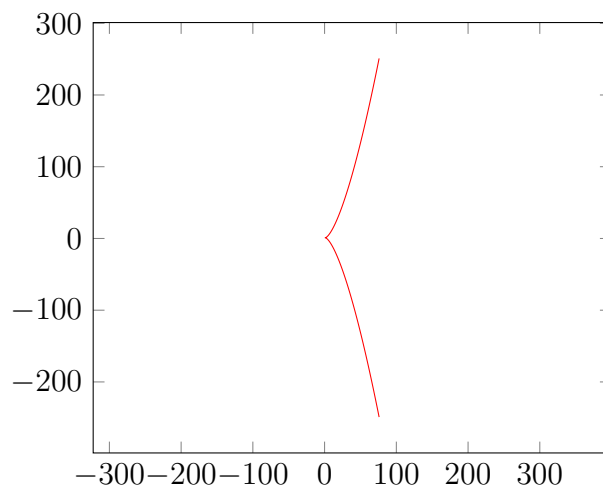
Since we were given the precondition $0 < t < \pi$, we only care about the interval:

$$\frac{\pi}{2} < t < \pi$$

Example 4

Find the equation of the tangent line at (4,3):

$$x = 3t^2 + 1 \quad y = 2t^3 + 1$$



The equation of the tangent line is of the form:

$$y - y_1 = m(x - x_1) \text{ where } m = \frac{dy}{dx}$$

$$\frac{dx}{dt} = 6t$$

$$\frac{dy}{dt} = 6t^2$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$= \frac{6t^2}{6t}$$

$$= t$$

Tangent line at (4,3):

$$\begin{aligned}
 y - y_1 &= \frac{dy}{dx}(x - x_1) \\
 y - y_1 &= t(x - x_1) \\
 3 - (2t^3 + 1) &= t(4 - (3t^2 + 1)) \\
 3 - 2t^3 - 1 &= t(4 - 3t^2 - 1) \\
 2 - 2t^3 &= 4t - 3t^3 - t \\
 t^3 - 3t + 2 &= 0 \\
 t^3 - t^2 + t^2 - t - 2t + 2 &= 0 \\
 t^2(t - 1) + t(t - 1) - 2(t - 1) &= 0 \\
 (t - 1)(t^2 + t - 2) &= 0 \\
 (t - 1)(t + 2)(t - 1) &= 0 \\
 (t - 1)^2(t - 2) &= 0 \\
 t = 1 \quad t = 2
 \end{aligned}$$

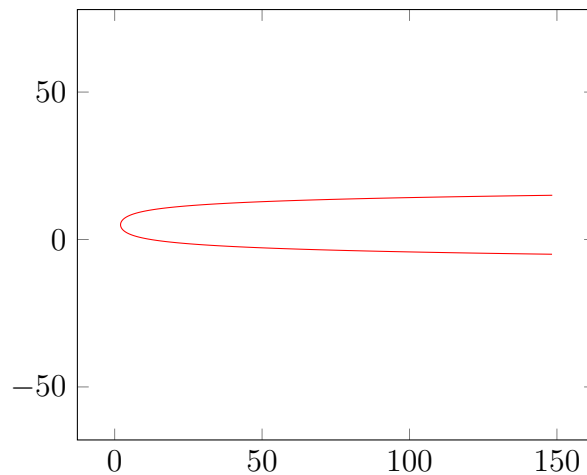
At $t = 1$, $(x, y) = (4, 3)$, therefore the equation of the tangent line is:

$$\begin{aligned}
 y - 3 &= 1(x - 4) \\
 y &= x - 1
 \end{aligned}$$

Example 5

Find the arc length of the curve:

$$x = e^t + e^{-t} \quad y = 5 - 2t \quad 0 \leq t \leq 3$$

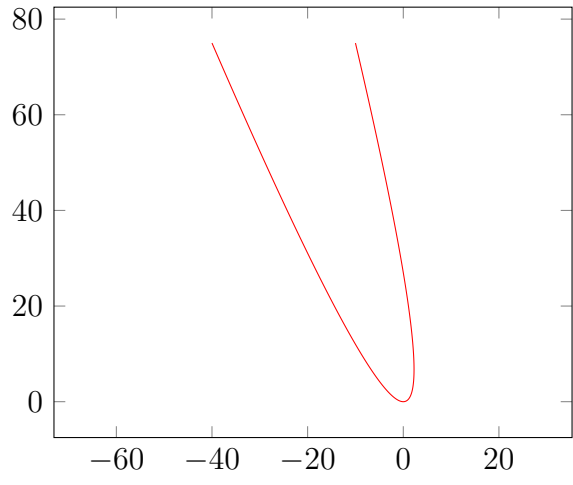


$$\begin{aligned}
L &= \int_a^b \sqrt{1 + (f'(x))^2} \, dx \\
f'(x) &= \frac{dy}{dx} = \frac{dy/dt}{dx/dt} \\
L &= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt \\
&= \int_0^3 \sqrt{(e^t - e^{-t})^2 + (-2)^2} \, dt \\
&= \int_0^3 \sqrt{e^{2t} + e^{-2t} - 2e^t e^{-t} + 4} \, dt \\
&= \int_0^3 \sqrt{e^{2t} + e^{-2t} + 2} \, dt \\
&= \int_0^3 \sqrt{(e^t + e^{-t})^2} \, dt \\
&= \int_0^3 (e^t + e^{-t}) \, dt \\
&= \left[e^t - e^{-t} \right]_0^3 \\
&= e^3 - e^{-3}
\end{aligned}$$

Example 6

Find the surface area of the solid of revolution:

$$x = 3t - t^2 \quad y = 3t^2 \quad 0 \leq t \leq 1$$



$$\begin{aligned}
S &= \int 2\pi y \, dS \\
&= \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx \\
&= 2\pi \int_a^b y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt \\
&= 2\pi \int_0^1 (3t^2) \sqrt{(3 - 2t^2)^2 + (6t)^2} \, dt \\
&= 6\pi \int_0^1 t^2 \sqrt{9 + 9t^4 - 12t^2 + 36t^2} \, dt \\
&= 6\pi \int_0^1 t^2 \sqrt{3^2 + (3t^2)^2 + 2(3)(3t^2)} \, dt \\
&= 6\pi \int_0^1 t^2 \sqrt{(3 + 3t^2)^2} \, dt \\
&= 6\pi \int_0^1 t^2 (3 + 3t^2) \, dt \\
&= 18\pi \int_0^1 (t^2 + t^4) \, dt \\
&= 18\pi \left[\frac{t^3}{3} + \frac{t^5}{5} \right]_0^1 \\
&= 18\pi \left(\frac{1}{3} + \frac{1}{5} \right) \\
&= 18\pi \left(\frac{8}{15} \right) \\
&= \frac{48\pi}{5}
\end{aligned}$$

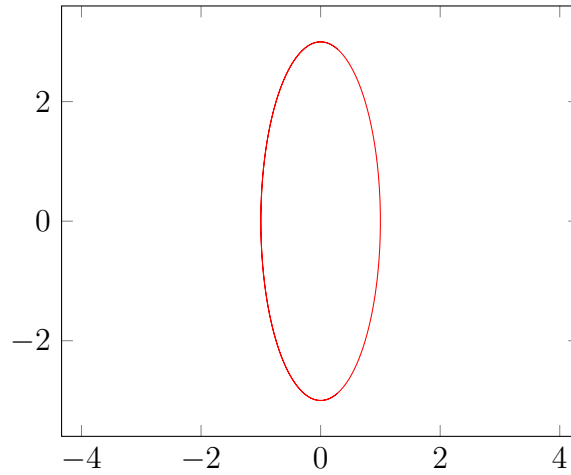
Example 7

$$x = a \cos(\theta) \quad y = b \sin(\theta) \quad 0 \leq \theta \leq 2\pi$$

$$\cos(\theta) = \frac{x}{a}$$

$$\sin(\theta) = \frac{y}{b}$$

$$\sin^2(\theta) + \cos^2(\theta) = \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



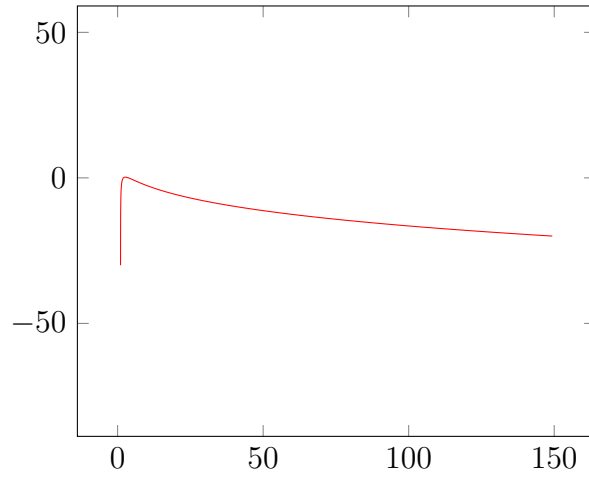
$$\begin{aligned}
 A &= 4 \int_a^b y \, dx \\
 &= 4 \int_{\pi/2}^0 (b \sin(\theta))(-a \sin(\theta)) \, d\theta \\
 &= -4ab \int_{\pi/2}^0 \sin^2(\theta) \, d\theta \\
 &= 4ab \int_0^{\pi/2} \frac{1}{2} - \frac{1}{2} \cos(2\theta) \, d\theta \\
 &= 2ab \int_0^{\pi/2} 1 - \cos(2\theta) \, d\theta \\
 &= 2ab \left[\theta - \frac{\sin(2\theta)}{2} \right]_0^{\pi/2} \\
 &= 2ab \left[\left(\frac{\pi}{2} - \frac{\sin(\pi)}{2} \right) - \left(0 - \frac{\sin(0)}{2} \right) \right] \\
 &= ab\pi
 \end{aligned}$$

Problem 33

Find the area under the curve bounded by:

$$x = 1 + e^t \quad y = t - t^2$$

and the x-axis:



$$A = \int_a^b y \, dx$$

$$y = 0 = t - t^2$$

$$t(1 - t) = 0$$

$$t = 0 \quad (x = 2) \quad t = 1 \quad (x = 1 + e)$$

$$A = \int_2^{1+e} y \, dx$$

$$= \int_{x=2}^{x=1+e} (t - t^2) \, dx$$

$$= \int_0^1 (t - t^2)e^t \, dt$$

$$= (t - t^2)e^t - \int (1 - 2t)e^t \, dt \quad (\text{Integration By Parts})$$

$$= t(1 - t)e^t - \left[(1 - 2t)e^t - \int -2e^t \, dt \right]$$

$$\int_0^1 (t - t^2)e^t \, dt = \left[t(1 - t)e^t - (1 - 2t)e^t - 2e^t \right]_0^1$$

$$= 3 - e$$

You can find all my notes at <http://omgimanerd.tech/notes>. If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech