Arc Length

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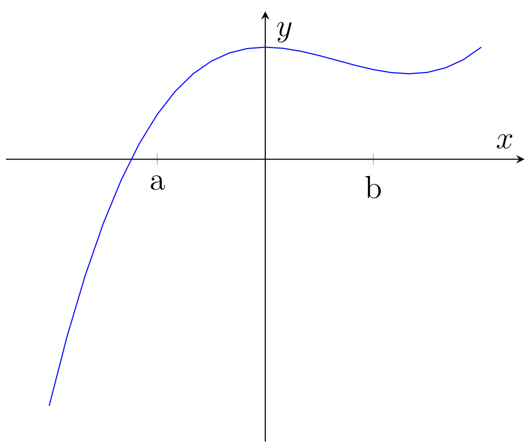
Arc Length



Suppose we want to find the distance between f(a) and f(b) along the curve.

$$L = \int_a^b \sqrt{1 + f'(x)^2}\ \mathrm{d}x$$

The arc length function is denoted:

$$S(x) = \int_a^x \sqrt{x + f'(t)^2}\ \mathrm{d}t$$

Example

$$f(x) = y = \ln(1 - x^2) \quad 0 \leq x \leq \frac{1}{2}$$

$$L = \int_a^b \sqrt{1 + f'(x)^2}\ \mathrm{d}x$$

$$f'(x) = \frac{0 - 2x}{1 - x^2} = \frac{-2x}{1 - x^2}$$

$$\begin{aligned}
1 + f'(x)^2 &= 1 + (\frac{-2x}{1 - x^2})^2 \\
&= 1 + \frac{4x}{(1 - x^2)^2} \\
&= \frac{(1 - x^2)^2 + 4x^2}{(1 - x^2)^2} \\
&= \frac{1 + x^4 - 2x^2 + 4x^2}{(1 - x^2)^2} \\
&= \frac{x^4 + 2x^2 + 1}{(1 - x^2)^2} \\
&= \frac{(1 + x^2)^2}{(1 - x^2)^2} \\
&= (\frac{1 + x^2}{1 - x^2})^2
\end{aligned}$$

Therefore:

$$\begin{aligned}
L &= \int_0^{\frac{1}{2}} \sqrt{1 + f'(x)^2}\ \mathrm{d}x \\
&= \int_0^{\frac{1}{2}} \frac{1 + x^2}{1 - x^2}\ \mathrm{d}x \\
&= \int_0^{\frac{1}{2}} \frac{2 - (1 - x^2)}{1 - x^2}\ \mathrm{d}x \\
&= \int_0^{\frac{1}{2}} \frac{2}{1 - x^2} - \frac{1 - x^2}{1 - x^2}\ \mathrm{d}x \\
&= \int_0^{\frac{1}{2}} \frac{2}{1 - x^2} - 1\ \mathrm{d}x \\
&= \int_0^{\frac{1}{2}} \frac{2}{(1 - x)(1 + x)} - 1\ \mathrm{d}x
\end{aligned}$$

By partial fractions:

$$\int_0^{\frac{1}{2}} \frac{A}{1-x} + \frac{B}{1+x} - 1 \text{ d}x$$

$$A = B = 1$$

$$\Big[-A\ln|1-x| + B\ln|1+x| - x \Big]_0^{\frac{1}{2}}$$

$$L = -\ln|\frac{1}{2}| + \ln|\frac{3}{2}| - \frac{1}{2}$$

You can find all my notes at `http://omgimanerd.tech/notes`. If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech