Trigonometric Integrals

Alvin Lin

Calculus II: August 2016 - December 2016

Trigonometric Integrals

Using concepts from Trigonometric Substitution, we can solve integrals with trigonometric functions by separating out even powers and using the following identities:

$$\sin^2(\theta) + \cos^2(\theta) = 1$$

$$1 + \tan^2(\theta) = \sec^2(\theta)$$
$$1 + \cot^2(\theta) = \csc^2(\theta)$$

If we can isolate an even power of sin, we can substitute:

$$\sin^2(\theta) = 1 - \cos^2(\theta)$$

If we can isolate an even power of cos, we can substitute:

$$\cos^2(\theta) = 1 - \sin^2(\theta)$$

The same rules apply for even powers of tan, sec, cot, and csc.

Practice Problem 2

$$\int \sin^3(\theta)\cos^4(\theta)\mathrm{d}\theta$$
$$\int \sin^2(\theta)\sin(\theta)\cos^3(\theta)\mathrm{d}\theta$$
$$Let : x = \cos(\theta)$$
$$\mathrm{d}x = -\sin(\theta)\mathrm{d}\theta$$

$$-\int (1-\cos^2(\theta))\cos^4(\theta)\mathrm{d}x$$

$$-\int (1-x^2)x^4\mathrm{d}x$$

$$-\int x^4 - x^6\mathrm{d}x$$

$$-\frac{1}{5}x^5 + \frac{1}{7}x^7 + C$$

$$= -\frac{1}{5}\cos^5(\theta) + \frac{1}{7}\cos^7(\theta) + C$$

Practice Problem 4

$$\int_0^{\pi/2} \sin^5(x)\mathrm{d}x$$

$$\int_0^{\pi/2} \sin^4(x)\sin(x)\mathrm{d}x$$

$$\int_0^{\pi/2} (1-\cos^2(x))^2\sin(x)\mathrm{d}x$$

$$\int_0^{\pi/2} (\cos^4(x) - 2\cos^2(x) + 1)\sin(x)\mathrm{d}x$$

$$Let: t = \cos(x)$$

$$\mathrm{d}t = -\sin(x)\mathrm{d}x$$

The limits from 0 to $\pi/2$ become 1 to 0. $\left(\int_0^{\pi/2} \to \int_1^0\right)$

$$-\int_1^0 t^4 - 2t^2 + 1\mathrm{d}t$$

We can flip the integral and negate it to remove the minus sign.

$$\int_0^1 t^4 - 2t^2 + 1\mathrm{d}t$$

$$\left[\frac{1}{5}t^5 - \frac{2}{3}t^3 + t\right]_0^1$$

$$\frac{1}{5} - \frac{2}{3} + 1 = \frac{3 - 10 + 15}{15}$$
$$= \frac{8}{15}$$

Practice Problem 13

$$\int \sqrt{cos(\theta)} \sin^3(\theta) \mathrm{d}\theta$$
$$\int \sqrt{cos(\theta)}(1 - \sin^2(\theta)) \sin(\theta) \mathrm{d}\theta$$
$$Let : t = \cos(\theta)$$
$$\mathrm{d}t = -\sin(\theta) \mathrm{d}\theta$$
$$-\int t^{\frac{1}{2}}(1 - t^2) \mathrm{d}t$$
$$\int -t^{\frac{1}{2}} + t^{\frac{5}{2}} \mathrm{d}t$$
$$-\frac{2}{3} t^{\frac{3}{2}} + \frac{2}{7} t^{\frac{7}{2}} + C$$
$$-\frac{2}{3} \cos^{\frac{3}{2}}(\theta) + \frac{2}{7} \cos^{\frac{7}{2}}(\theta) + C$$

Practice Problem 21

$$\int \tan(x) \sec^3(x) \mathrm{d}x$$
$$\int \tan(x) \sec(x) \sec^2(x) \mathrm{d}x$$
$$Let : t = \sec(x)$$
$$\mathrm{d}t = \sec(x) \tan(x) \mathrm{d}x$$
$$\int t^2 \mathrm{d}t$$
$$\frac{1}{3} t^3 + C$$
$$= \frac{1}{3} \sec^3(x) + C$$

Practice Problem 22

$$\int \tan^2(\theta)\sec^4(\theta)\mathrm{d}\theta$$

$$\int \tan^2(\theta)\sec^2(\theta)\sec^2(\theta)\mathrm{d}\theta$$

$$\int \tan^2(\theta)(1+\tan^2(\theta))\sec^2(\theta)\mathrm{d}\theta$$

$$Let : t = \tan(\theta)$$

$$\mathrm{d}t = \sec^2(\theta)$$

$$\int t^2(1+t^2)\mathrm{d}t$$

$$\int t^2 + t^6\mathrm{d}t$$

$$\frac{1}{3}t^3 + \frac{1}{5}t^5 + C$$

$$\frac{1}{3}\tan^3(\theta) + \frac{1}{5}\tan^5(\theta) + C$$

You can find all my notes at `http://omgimanerd.tech/notes`. If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech