

Integration By Parts

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Integration By Parts

Product Formula:

$$\frac{d}{dx}(fg) = fg' + f'g$$

Integrate on both sides:

$$fg = \int fg' dx + \int f'g dx$$

$$\int fg' dx = fg - \int f'g dx$$

Or with uv as variables:

$$\int u dv = uv - \int v du$$

For difficult integrals, this method can be used to rewrite the integral into a friendlier form.

Practice problem 1

$$\int x \cos(5x) dx$$

$$\text{Let : } f(x) = x \quad g'(x) = \cos(5x) dx$$

$$f'(x) = dx \quad g(x) = \frac{\sin(5x)}{5}$$

$$\int fg' dx = fg - \int f'g dx$$

$$\begin{aligned}
\int x \cos(5x) \, dx &= \frac{x \sin(5x)}{5} - \frac{1}{5} \int \sin(5x) \, dx \\
&= \frac{x \sin(5x)}{5} - \frac{1}{5} \left(-\frac{\cos(5x)}{5} \right) + C \\
&= \frac{1}{5} x \sin(5x) + \frac{1}{25} \cos(5x) + C
\end{aligned}$$

Practice problem 10

$$\begin{aligned}
&\int \ln(\sqrt{x}) \, dx \\
&\frac{1}{2} \int \ln(x) \, dx = \frac{1}{2} \int 1 \ln(x) \, dx \\
&\text{Let : } g'(x) = 1 \, dx \quad f(x) = \ln(x) \\
&\quad g(x) = x \quad f'(x) = \frac{1}{x} \, dx \\
&\int \ln(\sqrt{x}) \, dx = \frac{1}{2} \left(x \ln(x) - \int x \frac{1}{x} \, dx \right) \\
&\quad \frac{1}{2} \left(x \ln(x) - \int 1 \, dx \right) \\
&\quad = \frac{x \ln(x) - x}{2} + C
\end{aligned}$$

Practice problem 12

$$\begin{aligned}
&\int \tan^{-1} 2u \, du \\
&\int \tan^{-1} 2u \, du = \int 1 \tan^{-1} 2u \, du \\
&\text{Let : } g'(u) = 1 \, du \quad f(u) = \tan^{-1} 2u \\
&\quad g(u) = u \quad f'(u) = 2 \frac{1}{1+4u^2} \, du \\
&\int \tan^{-1} 2u \, du = \left(u \tan^{-1} 2u \right) - \int \frac{2u}{1+4u^2} \, du
\end{aligned}$$

After using integration by parts, we can solve the new integral with u -substitution.

$$\begin{aligned} & \int \frac{2u}{1+4u^2} du \\ \text{Let : } & x = 1 + 4u^2 \\ & dx = 8u du \\ & \int \frac{2u}{x} \frac{dx}{8u} \\ & \frac{1}{4} \int \frac{dx}{x} \\ & \frac{1}{4} \ln(x) + C \\ & = \frac{1}{4} \ln(1 + 4u^2) + C \end{aligned}$$

Therefore:

$$\begin{aligned} & \int \tan^{-1} 2u du \\ & = u \tan^{-1} 2u - \frac{1}{4} \ln(1 + 4u^2) + C \end{aligned}$$

Practice problem 32

$$\begin{aligned} & \int_1^2 \frac{(\ln(x))^2}{x^3} dx \\ \text{Let : } & u = \ln(x) \\ & e^u = x \end{aligned}$$

The limit of the integral from 1 to 2 changes to e^1 to e^2

$$\begin{aligned} & e^u du = dx \\ & \int_e^{e^2} \frac{u^2 e^u du}{(e^u)^3} \\ & = \int_e^{e^2} u^2 e^{-2u} du \end{aligned}$$

Now we can use integration by parts:

$$\begin{aligned}
 \text{Let : } f(x) &= u^2 \quad g'(x) = e^{-2u} \\
 f'(x) &= 2u \, du \quad g(x) = \frac{e^{-2u}}{-2} \\
 \int_e^{e^2} u^2 e^{-2u} \, du &= \left(u^2 \frac{e^{-2u}}{-2} \right) - \int 2u \frac{e^{-2u}}{-2} \, du \\
 &= \frac{u^2 e^{-2u}}{-2} + \int u e^{-2u} \, du
 \end{aligned}$$

In order to solve the integral properly, you need to use integration by parts again:

$$\begin{aligned}
 \text{Let : } f(x) &= u \quad g'(x) = e^{-2u} \, du \\
 f'(x) &= 1 \, du \quad g(x) = \frac{e^{-2u}}{-2} \\
 \frac{u^2 e^{-2u}}{-2} + \left(\frac{u^2 e^{-2u}}{-2} - \int (1) \frac{e^{-2u}}{-2} \, du \right) \\
 \frac{u^2 e^{-2u}}{-2} + \left(\frac{u^2 e^{-2u}}{-2} + \frac{1}{2} \int e^{-2u} \, du \right) \\
 -u^2 e^{-2u} + \left(\frac{1}{2} \int e^{-2u} \, du \right) \\
 \left[-u^2 e^{-2u} + \frac{1}{2} \left(\frac{e^{-2u}}{-2} \right) \right]_e^{e^2} \\
 -e^4 e^{-2e^2} - \frac{e^{-2e^2}}{4} - \left[-e^2 e^{-2e} - \left(\frac{e^{-2e}}{4} \right) \right] \\
 -e^4 e^{-2e^2} - \frac{e^{-2e^2}}{4} + e^2 e^{-2e} + \frac{e^{-2e}}{4} \\
 = -e^{-8e^2} + e^{-4e} + \frac{e^{-2e^2+e^{-2e}}}{4}
 \end{aligned}$$

Practice problem 38

$$\int \cos(\ln(x)) \, dx$$

$$\text{Let : } u = \ln(x)$$

$$x = e^u$$

$$dx = e^u \, du$$

$$\int \cos(u)e^u \, du = \int e^u \cos(u) \, du$$

We can describe this integral in a more general form as:

$$I = \int e^{ax} \cos(bx) \, dx$$

where a and b are both 1. Using integration by parts where:

$$f(x) = e^{ax} \quad g'(x) = \cos(bx) \, d(x)$$

we can get:

$$\frac{e^{ax} \sin(bx)}{b} - \int \frac{ae^{ax} \sin(bx)}{b} \, dx$$

$$\frac{e^{ax} \sin(bx)}{b} - \frac{a}{b} \int e^{ax} \sin(bx) \, dx$$

Using integration by parts again:

$$\frac{e^{ax} \sin(bx)}{b} - \frac{a}{b} \left[e^{ax} \left(\frac{-\cos(bx)}{b} \right) - \int ae^{ax} \left(\frac{-\cos(bx)}{b} \right) \, dx \right]$$

$$I = \int e^{ax} \cos(bx) \, dx = \frac{e^{ax} \sin(bx)}{b} + \frac{a}{b^2} e^{ax} \cos(bx) - \frac{a^2}{b^2} \int e^{ax} \cos(bx) \, dx$$

Rewriting this:

$$\int e^{ax} \cos(bx) \, dx + \frac{a^2}{b^2} \int e^{ax} \cos(bx) \, dx = \frac{e^{ax}}{b^2} [b \sin(bx) + a \cos(bx)]$$

$$= \left(1 + \frac{a^2}{b^2} \right) \int e^{ax} \cos(bx) \, dx = \frac{e^{ax}}{b^2} [b \sin(bx) + a \cos(bx)]$$

$$(b^2 + a^2) \int e^{ax} \cos(bx) \, dx = e^{ax} [b \sin(bx) + a \cos(bx)]$$

$$I = \int e^{ax} \cos(bx) \, dx = \frac{e^{ax}}{a^2 + b^2} [b \sin(bx) + a \cos(bx)]$$

We can apply this general form back to the original integral.

$$\int e^u \cos(u) \, du$$

For this case:

$$\begin{aligned} a &= 1 & b &= 1 \\ \int e^u \cos(u) \, du &= \frac{e^x}{1+1} [\sin(x) + \cos(x)] \\ &= \frac{e^x(\sin(x) + \cos(x))}{2} \end{aligned}$$

Practice Problem 39

$$\int \theta^3 \cos(\theta^2) \, d\theta$$

First we start with substitution:

$$\begin{aligned} \text{Let : } \theta^2 &= x \\ 2\theta &= \frac{dx}{d\theta} & \frac{dx}{2} &= \theta \, d\theta \\ \int \theta^2 \cos(\theta^2) \theta \, d\theta &= \frac{1}{2} \int x \cos(x) \, dx \end{aligned}$$

Then we can use integration by parts:

$$\begin{aligned} &\int x \cos(x) \, dx \\ \text{Let : } u &= x & dv &= \cos(x) \, dx \\ du &= dx & v &= \sin(x) \\ x \cos(x) \, dx &= x \sin(x) - \int \sin(x) \, dx \\ &= x \sin(x) - \cos(x) + C \end{aligned}$$

Now we can substitute this back:

$$\begin{aligned} \frac{1}{2} \int x \cos(x) \, dx &= \frac{1}{2} x \sin(x) - \cos(x) + C \\ &= \frac{1}{2} \theta^2 \sin(\theta^2) - \cos(\theta^2) + C \end{aligned}$$

Practice Problem 44

$$\int x^{\frac{3}{2}} \ln(x) \, dx$$

$$\text{Let : } u = \ln(x) \quad dv = x^{\frac{3}{2}} \, dx$$

$$du = \frac{1}{x} \, dx \quad v = \frac{x^{\frac{5}{2}}}{\frac{5}{2}}$$

$$\int x^{\frac{3}{2}} \ln(x) \, dx = \frac{x^{\frac{5}{2}} \ln(x)}{\frac{5}{2}} - \int \frac{1}{x} \frac{x^{\frac{5}{2}}}{\frac{5}{2}} \, dx$$

$$\frac{2}{5} \ln(x) x^{\frac{5}{2}} - \frac{2}{5} \int x^{\frac{3}{2}} \, dx$$

$$\frac{2}{5} \ln(x) x^{\frac{5}{2}} - \frac{2}{5} \frac{x^{\frac{5}{2}}}{\frac{5}{2}} \, dx$$

You can find all my notes at <http://omgimanerd.tech/notes>. If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech