Intro to Computer Science Theory: Homework 3

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Problem 1

1. Consider the following languages, each defined over the alphabet $\{a, b\}$

   (a) $L_1 = \{x \mid |x| \geq 3 \text{ AND its third symbol from the right is } b\}$

   (b) $L_2 = \{x \mid x \text{ contains at least two } b\text{s OR at most one } a\}$

   (c) $L_3 = \{x \mid \text{starts with } a \text{ AND has even length, OR starts with } b \text{ AND has odd length}\}$

   Each of these languages is "complex," in the sense that it can be broken into the union or intersection of simpler languages, where each simpler language is defined by one of the clauses of the set builder between "OR" or "AND." For instance, the first set can be redefined as

$$L_1 = \{x \mid |x| \geq 3\} \cap \{x \mid x\text{'s third symbol from the right is } b\}$$

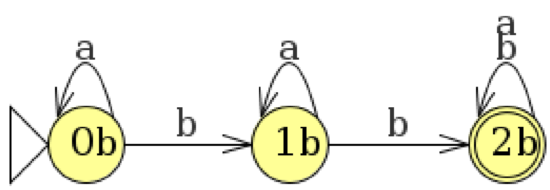
   (a) [2 points each] For L_2 and L_3 above, rewrite each of the definitions by removing all AND's and OR's and replacing them with the intersection or union of simpler sets based on each clause (as in the example for L_1 given above). Note that for L_3 this means you need four simple sets and parentheses to make your definition clear.

$$L_2 = \{x \mid x \text{ contains at least two } b\text{s}\} \cup \{x \mid x \text{ contains at most one } a\}$$

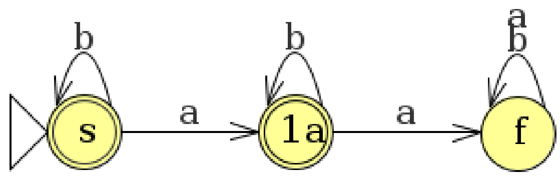
$$L_3 = \Big(\{x \mid x \text{ starts with } a\} \cap \{x \mid x \text{ has even length}\}\Big) \cup \Big(\{x \mid x \text{ starts with } b\} \cap \{x \mid x \text{ has odd length}\}\Big)$$

   (b) [1 point for each simple language, 8 points total] For each of the simple sets (two simple sets for L_1 and L_2, four for L_3), provide the state transition diagram for each. Label the states for each machine.

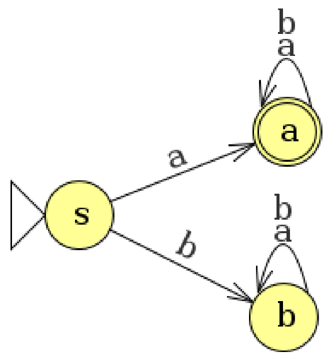
$\{x \mid x \text{ contains at least two } b\text{s}\}$



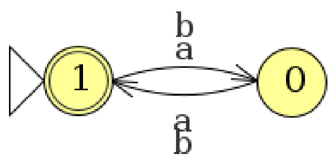
$\{x \mid x \text{ contains at most one } a\}$



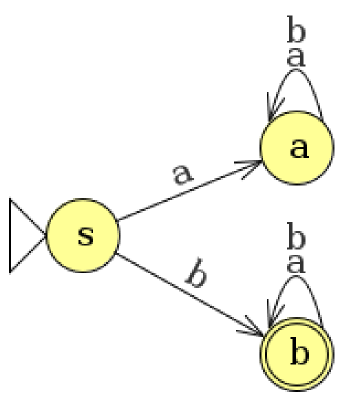
$\{x \mid x \text{ starts with } a\}$



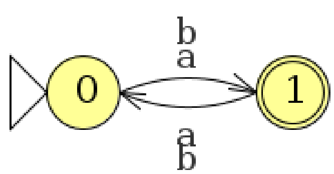
$\{x \mid x \text{ has even length}\}$



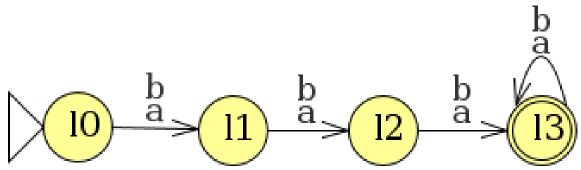
$\{x \mid x \text{ starts with } b\}$



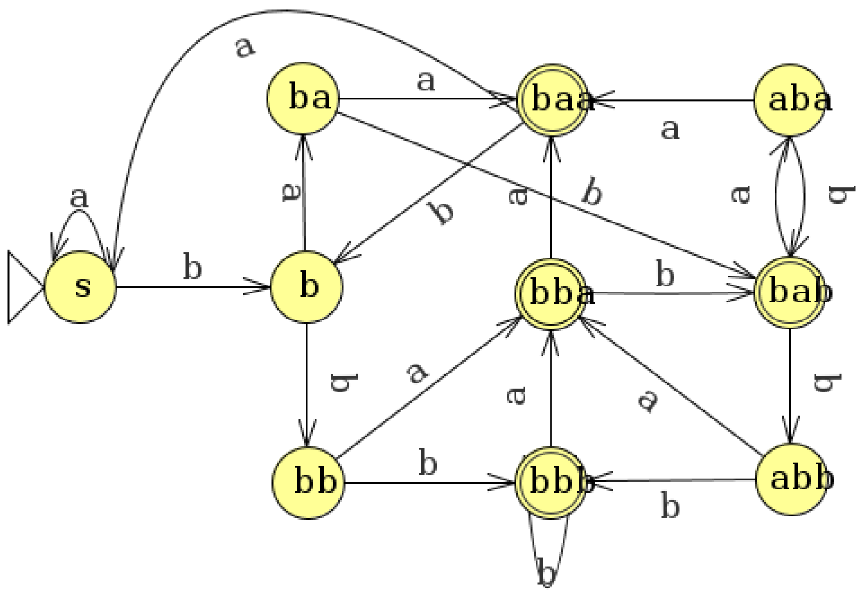
$\{x \mid x \text{ has odd length}\}$



$\{x \mid |x| \geq 3\}$



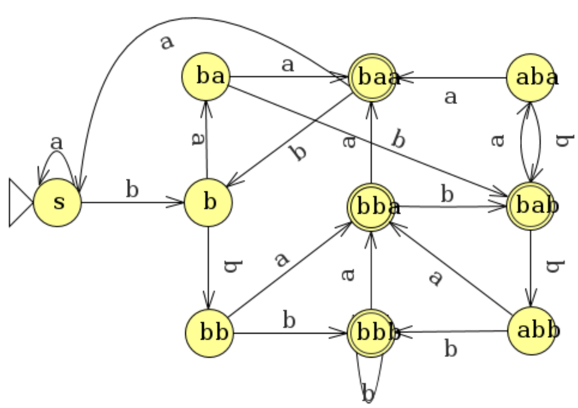
$\{x \mid x\text{'s third symbol from the right is } b\}$



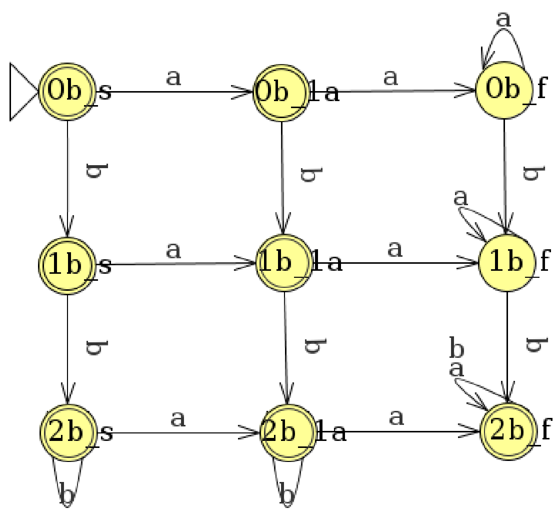
(c) [4 points for each complex language, 12 points total] For each of the three complex languages L_1, L_2, and L_3, use the construction from Theorem 1.25 (and the discussion in the footnote below) to construct FAs for each from the simple sets. You need only draw the state transition diagrams for each complex language, but the state names, accepting states etc. must reflect the construction. Note also that for L_1 and L_2 you need apply the construction once, but for L_3 you need to apply it three times.

L_1

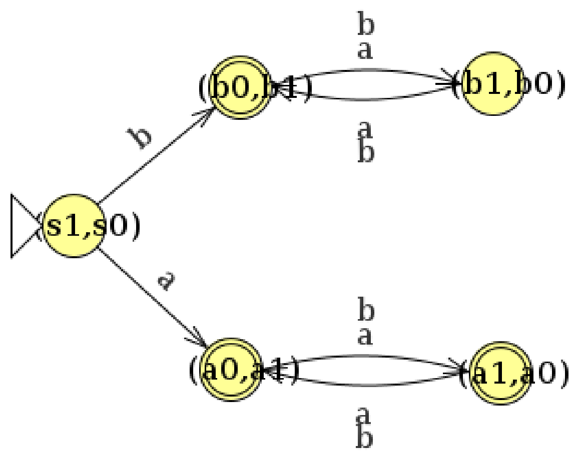
For L_1 the language $\{x \mid |x| \geq 3\}$ is a subset of the language $\{x \mid x\text{'s third symbol from the right is } b\}$, thus the state transition diagram for $\{x \mid x\text{'s third symbol from the right is } b\}$ is also the state transition diagram for L_1.



L_2



L_3



2. For any alphabet Σ, consider the following (recursive) definition of $|\cdot| : \Sigma^* \to \mathbb{N}$ (where $\mathbb{N}$ is the natural numbers, PLUS 0), on input $x \in \Sigma^*$ and $y \in \Sigma$ as

$$|x| = \begin{cases} 0 & \text{if } x = \epsilon \\ |y| + 1 & \text{otherwise, where } x = ya \wedge y \in \Sigma^* \wedge a \in \Sigma. \end{cases}$$

(a) [3 points] Write the following statement in quantifiable logic: "For any alphabet Σ, $z, w \in \Sigma^*$, and $b \in \Sigma$, if $|zw| = |z| + |w|$ then $|zwb| = |z| + |w| + 1$."

$$\forall \Sigma \Big(\forall z \forall w \forall b (z, w \in \Sigma^* \wedge b \in \Sigma \wedge |zw| = |z| + |w| \to |zwb| = |z| + |w| + 1) \Big)$$

(b) [3 points] Finish the proof. For each quantifier, you must use one of the proof strategies from Velleman. Use square brackets and logical forms to cite each use of a strategy, as in page 27–28 of the course notes.

i. [The statement is of the form $\forall x P(x)$] Let Σ be an arbitrary alphabet.

ii. [The statement is of the form $\forall b P(b)$] Choose an arbitrary b.

iii. [The statement is of the form $R \to S$] Suppose that $\forall z \forall w (z, w \in \Sigma^* \wedge b \in \Sigma \wedge |zw| = |z| + |w|$, we need to show that $|zwb| = |z| + |w| + 1$.

iv. By the definition $|x| = |y| + 1$ where $x = ya \wedge y \in \Sigma^* \wedge a \in \Sigma$, it follows that $|zwb| = |zw| + 1$.

v. From what is given, $\forall z \forall w |zw| = |z| + |w|$, it follows that $|zwb| = |z| + |w| + 1$

If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech