Intro to Computer Science Theory: Homework 1

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Problem 1

Analyze the logical forms of the following statements. Use the same level of detail as presented in page 18 of the course notes. Make sure that you add quantifiers to ensure that all the variables are bound. See also exercise 2 of section 2.1 of Velleman (2 points each).

1. I've given you all I've got.
$$\forall x(g(x) \to G(x))$$
where $g(x)$ stands for "I've got x" and $G(x)$ stands for "I've given you x".

2. Someday my prince will come.
$$\exists d P(d)$$
where $P(d)$ stands for "My prince will come on day d".

3. Nobody loves me like my baby.
$$\neg\exists x L(x)$$
where $L(x)$ stands for "x loves me like my baby".

4. If Fred can't do it, no one can.
$$\neg D(f) \to (\neg\exists x D(x))$$
where $D(x)$ stands for "x can do it" and f stands for Fred.

5. If Fred can't do it, everyone else can.
$$\neg D(f) \to (\forall x D(x))$$
where $D(x)$ stands for "x can do it" and f stands for Fred.

Problem 2

Let $A = \{z \in \mathbb{R} \mid 0 < z < 1\}$ and $B = \{z \in \mathbb{R} \mid 0 \le z \le 1\}$. Say whether each of the following statements is true or false (1 point each).

1. $\forall x \in A(\exists y \in A(x > y))$ True

2. $\forall x \in A(\exists y \in A(x \ge y))$ True

3. $\exists y \in A(\forall x \in A(x > y))$ False

4. $\exists y \in A(\forall x \in A(x \ge y))$ False

5. $\forall x \in B(\exists y \in B(x > y))$ False
6. $\forall x \in B(\exists y \in B(x \geq y))$ True
7. $\exists y \in B(\forall x \in B(x > y))$ False
8. $\exists y \in B(\forall x \in B(x \geq y))$ True

Problem 3

Prove that, for any language L, if $L \circ L \subseteq L$ then $L \circ L \circ L \subseteq L$, in the following steps.

1. Rewrite in quantifiable logic form, with no free variables, using $\subseteq$ in the writeup. (4 points)

$$\forall L(L \circ L \subseteq L \to L \circ L \circ L \subseteq L)$$

2. Substitute for each occurrance $\subseteq$ the definition of $\subseteq$. (4 points)

$$\forall L(\forall x(x \in (L \circ L) \to x \in L) \to \forall x(x \in (L \circ L \circ L) \to x \in L))$$

3. Continue the proof. For each quantifier, you must use one of the proof strategies from Velleman. Use square brackets and logical forms to cite each use of a strategy, as in page 27 of the course notes. (4 points)

   - Choose an arbitrary L [Velleman p.108]:

   $$\forall x(x \in (L \circ L) \to x \in L) \to \forall x(x \in (L \circ L \circ L) \to x \in L)$$

   - This is of the form $R \to S$ [Velleman p.87], assume

   $$\forall x(x \in (L \circ L) \to x \in L)$$

   and prove:

   $$\forall x(x \in (L \circ L \circ L) \to x \in L)$$

   - Choose an arbitrary x and y [Velleman p.108]:

   $$x \in (L \circ L) \to x \in L$$

   $$x \in (L \circ L \circ L) \to y \in L$$

   - This is of the form $R \to S$. Assume that $x \in L \circ L \circ L$. We need to show that $x \in L$.
   - By the definition of $\circ$, there exists $y \in L \circ L$ and $z \in L$ such that $x = yz$.
   - From our given, if $y \in L \circ L$, then it follows that $y \in L$.

If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech