Introduction to Computer Science Theory

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Pumping Lemmas

A general criteria for establishing non-regularity: If A is a regular language, then there is an integer p such that for any $s \in A$ with $|s| \geq p$ there are strings x, y, z such $s = xyz$ and:

1. for any $i \geq 0, xy^iz \in A$
2. $|y| > 0$
3. $|xy| \leq p$

This is called a pumping lemma because you repeat or pump the substring y.

Using the pumping lemma

The language $L = \{a^jb^{2j} \mid j \geq 0\}$ is not regular. To prove this, suppose for a contradiction that L is regular. Then by the pumping lemma, let p be the integer such that every string s in L with $|s| \geq p$ can be written as xyz, where $|xy| \leq p, |y| > 0$, and $xy^iz \in L$ for all $i \geq 0$.

- Let $s = a^pb^{2p}$. Then $s \in L$ and $|s| \geq p$. So, there exist strings x, y, z such that $s = xyz, |xy| \leq p, |y| > 0$, and $xy^iz \in L$ for all $i \geq 0$.
- In particular, $xz \in L$ (choosing $i = 0$)
- Since $|xy| \leq p$ and $|y| > 0$, y consists of 1 or more a's. If we remove y from a^pb^{2p}, we get a string with $2p$ b's and less than p a's.

- xz has $2p$ b's and less than p a's. But then, by the definition of L, $xz \notin L$. This is a contradiction.
- It follows that the assumption that L is regular is wrong.
- So, we have shown that L is not regular.

Example

Prove L is not regular.

$$L = \{a^{p^2} \mid p \geq 0\}$$

Assume that L is regular. By the pumping lemma, there is a $p \in \mathbb{N}$ such that for all $s \in L$ such that $|s| \geq p$ there exists x, y, z such that $s = xyz, |xy| \leq p, |y| > 0$ for all $i \geq 0, xy^iz \in L$. Let $s = a^{p^2}$. So by definition of L, $s \in L$ and $|s|$ is clearly greater than or equal to p. So, there exist x, y, z such that $s = xyz, |xy| \leq p, |y| > 0$ and for all $i \geq 0, xy^iz \in L$. Let $i = 2$.

By the pumping lemma, $xy^2z \in L$. Since $|y| > 0, |xy^2z| > p^2$. Thus $|xy^2z| \geq (p+1)^2$. Since $|xy| \leq p, |xy^2z| \leq p^2 + p$.

So, $(p+1)^2 \leq p^2 + p$. This is a contradiction and thus L is not regular.

Example

$$\Sigma = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

$$L = \{x \in \Sigma^* \mid \text{the top string is the reverse of the bottom}\}$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \in L$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \in L$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \notin L$$

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Assume that L is regular. By the pumping lemma, there exists $p \in \mathbb{N}$ such that for all $s \in L$ such that $|s| \geq p$ there exist x, y, z such that $s = xyz, |xy| \leq p, |y| > 0$, and $\forall i \geq 0 (xy^iz \leq L)$.

Let $s = \begin{bmatrix} 0 \\ 1 \end{bmatrix}^p \begin{bmatrix} 1 \\ 0 \end{bmatrix}^p$. Let x, y, z be such that $s = xyz$, $|xy| \leq p$, $|y| > 0$ and for all $i \geq 0, xy^iz \in L$.

Let $i = 3$. By the pumping lemma, $xy^3z \in L$. But $xy^3z = \begin{bmatrix} 0 \\ 1 \end{bmatrix}^{p+2|y|} \begin{bmatrix} 1 \\ 0 \end{bmatrix}^p$. This is a contradiction and thus L is not regular.

You can find all my notes at `http://omgimanerd.tech/notes`. If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech